Warmup Q6)

Abelian group G is generated by w, x, y, z with relations

$$\begin{cases} 3w + 5x - 3y & = 0 \\ 4w + 2x \quad\quad - 2z & = 0 \end{cases}$$

What is G isomorphic to? (Rank and torsion coefficients?)

Hint

Reduce

$$\begin{pmatrix} 3 & 5 & -3 & 0 \\ 4 & 2 & 0 & -2 \end{pmatrix}$$

with integer row and column operations. (Why?)

Fund Thm for Finitely Generated Abelian Groups

Theorem (FTFGAG, 12.13)

Every finitely generated abelian group can be written as

$$\mathbb{Z}/d_1 \times \cdots \times \mathbb{Z}/d_k \times \mathbb{Z}^r \, .$$

Moreover can choose $d_1 > 1$ and $d_i \mid d_{i+1}$, then this form is unique.

In the case where $d_1 > 1$, and $d_i \mid d_{i+1}$

Definition (12.14)

- The rank is r
- The torsion coefficients are $d_1, \ldots, d_k$. (With multiplicities.)

Putting abelian group into FTFGAG form

Generators $x_1, \ldots, x_n$ and relations $R_i\colon a_{i1}x_1 + \cdots + a_{in}x_n = 0$.
Make matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Apply integer row and column operations to reduce to diagonal

$$D = \begin{pmatrix} d_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & d_k & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \underbrace{0 & \cdots & 0}_{r \text{ columns}} \end{pmatrix} \Big\} \text{Can pad with rows of 0's}$$

So group is $\cong \mathbb{Z}/d_1 \times \cdots \times \mathbb{Z}/d_k \times \mathbb{Z}^r$. (Note: $r =$ rank of $\ker D$)

Row and column operations

Generators $x_1, \dots, x_n$ and relations $R_i\colon a_{i1}x_1 + \cdots + a_{in}x_n = 0$.
Use integer row and column operations

- Swap row i and row j — Relabel $R_i \leftrightarrow R_j$
- Multiply a row i by -1 — Replace R_i with inverse $-R_i$
- Add $\alpha\times$ row i to row j, $\alpha \in \mathbb{Z}$ — Replace R_j with $R_j + \alpha R_i$

- Swap col i and col j — Relabel $x_i \leftrightarrow x_j$
- Multiply a col i by -1 — Substitute x_i with $x_i = -x_i'$
- Add $\alpha\times$ col i to col j, $\alpha \in \mathbb{Z}$ — Substitute $x_i = x_i' + \alpha x_j$

Q6 - What finitely generated abelian group is this?

Generators w, x, y, z, relations

$$\begin{cases} 3w + 5x - 3y = 0 \\ 4w + 2x - 2z = 0 \end{cases}$$

$$\begin{pmatrix} 3 & 5 & -3 & 0 \\ 4 & 2 & 0 & -2 \end{pmatrix} \quad (\text{columns } w, x, y, z)$$

Sub $z = z' + 2w$ $\qquad c_1 \mapsto c_1 + 2c_4$

$\cong$ Generators w, x, y, z', relations

$$\begin{cases} 3w + 5x - 3y = 0 \\ 2x - 2z' = 0 \end{cases}$$

$$\sim \begin{pmatrix} 3 & 5 & -3 & 0 \\ 0 & 2 & 0 & -2 \end{pmatrix} \quad (\text{columns } w, x, y, z')$$

Sub $z' = z'' + x$ $\qquad c_2 \mapsto c_2 + c_4$

$\cong$ Generators w, x, y, z'', relations

$$\begin{cases} 3w + 5x - 3y = 0 \\ - 2z'' = 0 \end{cases}$$

$$\sim \begin{pmatrix} 3 & 5 & -3 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \quad (\text{columns } w, x, y, z'')$$

Sub $y = y' + w + 2x$	$c_1 \mapsto c_1 + c_3$, $c_2 \mapsto c_2 + 2c_3$
$\cong$ Generators w, x, y', z'', relations $\begin{cases} -x - 3y' = 0 \\ -2z'' = 0 \end{cases}$	$\sim \begin{pmatrix} 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$ (columns w, x, y', z'')
Sub $x = x' - 3y'$	$c_3 \mapsto c_3 - 3c_2$
$\cong$ Generators w, x', y', z'', relations $\begin{cases} -x' = 0 \\ -2z'' = 0 \end{cases}$	$\sim \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$ (columns w, x', y', z'')
Replace R_1, R_2 by inverses	$r_1 \mapsto -r_1$, $r_2 \mapsto -r_2$
$\cong$ Generators w, x', y', z'', relations $\begin{cases} x' = 0 \\ 2z'' = 0 \end{cases}$	$\sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ (columns w, x', y', z'')

Relabel $w \leftrightarrow x'$, then $x' \leftrightarrow z''$

Generators w, x', y', z'', relations

$$\cong \quad \begin{cases} w & = 0 \\ \quad 2x' & = 0 \end{cases}$$

Swap $c_1 \leftrightarrow c_2$, then $c_2 \leftrightarrow c_4$

$$\sim \begin{pmatrix} \overset{w}{1} & \overset{x'}{0} & \overset{y'}{0} & \overset{z''}{0} \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

- Now it's easy to identify the group

$$\begin{aligned} G &\cong \frac{\mathbb{Z}\langle w \rangle}{\langle w \rangle} \times \frac{\mathbb{Z}\langle x' \rangle}{\langle 2x' \rangle} \times \mathbb{Z}\langle y' \rangle \times \mathbb{Z}\langle z'' \rangle \\ &\cong \mathbb{Z}/1 \quad \times \mathbb{Z}/2 \quad \times \mathbb{Z} \quad \times \mathbb{Z} \\ &\cong \mathbb{Z}/2 \times \mathbb{Z}^2 \end{aligned}$$

- Torsion coefficients $d_1 = 2$, and rank $r = 2$