Analysis 1

Some extra, or interesting, or challenging questions

See `http://www.maths.dur.ac.uk/users/steven.charlton/analysis1_1314` for updates.

Below are come extra, or interesting, or challenging questions related to, or extending, the concepts and results covered in the Analysis 1 course.

Questions marked with:

- $*$ are tricky, but require nothing more than what is covered in the course
- # are more difficult – they may require some clever insight, or trick
- $\Rightarrow$ require using results from a previous question, so may exceed the scope of the course

1 Limit Computations

*Q1) Calculate $\lim_{n\to\infty} x_n$ for the following sequences:

i) $x_n = \sqrt[3]{n^3 + 2n^2 + 7n - 3} - \sqrt[3]{n^3 - 3n^2 - 6n + 4}$.

Hint: Make use of $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

ii) More generally $x_n = \sqrt[3]{n^3 + an^2 + bn + c} - \sqrt[3]{n^3 + dn^2 + en + f}$

Q2) Use the squeezing theorem to evaluate $\lim_{n\to\infty} x_n$ for the following sequences:

i) $x_n = \dfrac{n^2 + \cos(n/5)}{2n^2 - \sin(n^2)}$

*ii) $x_n = \left(\dfrac{3\cos n + 4\sin n}{6}\right)^n$

Hint: Can you write $3\cos n + 4\sin n$ in the form $R\sin(n + \theta)$?

iii) $x_n = (\cosh(n))^{1/n}$

Hint: Squeeze $\log x_n$.

*Q3) Given *Stirling's approximation* $\lim_{n\to\infty} \dfrac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$, compute $\lim_{n\to\infty} x_n$ for the following sequences:

i) $x_n = \dfrac{n(3n)!}{3^{3n}(n!)^3}$

ii) $x_n = \dfrac{2^{4n}}{n\binom{2n}{n}^2}$

2 ϵ-N Proofs of Limit Computations

Q1) Evaluate $\lim_{n\to\infty} x_n$, and then give a formal ϵ-N proof of this result, for the following sequences:

i) $x_n = \dfrac{n+5}{n^2+3}$

ii) $x_n = \dfrac{2n^2+7}{n^2+2n+1}$

iii) $x_n = \dfrac{3n^3-2n+7}{n^4+7n+1}$

iv) $x_n = \dfrac{3+2n^2}{2-n+n^2}$

*Q2) Evaluate $\lim_{n\to\infty} x_n$, and then give a formal ϵ-N proof of this result, for the following sequences:

i) $x_n = \sqrt{\dfrac{2+x}{1+x}}$

ii) $x_n = \dfrac{3\cos(n)+2n^2}{2-n+n^2}$

iii) $x_n = \dfrac{3+2n^2}{2-n\sin(n)+n^2}$

3 Limit Theorems

*Q1) **COLT:** If $x_n \to L$ and $y_n \to M$ as $n \to \infty$, use the ϵ-N definition of limits to prove that as $n \to \infty$:

i) $x_n y_n \to LM$

Hint: Rewrite $x_n y_n - LM$ as $(x_n - L)(y_n - M) + L(y_n - M) + M(x_n - L)$.

ii) $\dfrac{1}{y_n} \to \dfrac{1}{M}$ (assuming M and all y_n are non-zero)

*Q2) If $a_n \to L$ as $n \to \infty$, give direct ϵ-N proofs of the following:

i) $a_n^2 \to L^2$ (don't just appeal to calculus of limits)

ii) $\sqrt{a_n} \to \sqrt{L}$ (assuming $a_n \geq 0$ for all n)

iii) $\log a_n \to \log L$ (assuming $a_n > 0$ for all n, and $L > 0$)

iv) $\exp(a_n) \to \exp(L)$

#Q3) **Cesàro Mean:** Suppose that the sequence $x_n \to L$ as $n \to \infty$. Consider the sequence

$$y_n \coloneqq \frac{1}{n}(x_1 + x_2 + \cdots + x_n)\,.$$

Prove that $\lim_{n\to\infty} y_n = L$, as well.

Hint: Since $x_n \to L$, find N_0 such that $n \geq N_0$ implies $|x_n - L| < \epsilon/2$. Split up the sum in y_n at the fixed number N_0.

⇒Q4) Use the result in the previous question to evaluate $\lim_{n\to\infty} y_n$ for the following sequences:

i) $y_n = \dfrac{1}{n} \log\left(1^{1/1}\, 2^{1/2}\, 3^{1/3} \, \cdots \, n^{1/n}\right)$

Can you evaluate any of these more directly?

$^{\#}$Q5) Suppose a_n is a sequence such that $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = L$. Prove that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, as well. Does the converse hold?

Note: This says that whenever the ratio test succeeds (reaches a conclusion on convergence or divergence) then the root test (see Section 5, Q1) below) will also succeed (and reach the same conclusion, of course). So the root test is *stronger* than the ratio test.

$^{\Rightarrow}$Q6) Use the result in the previous question to evaluate $\lim_{n\to\infty} x_n$ for the following sequences:

i) $x_n = \sqrt[n]{\dfrac{1}{n!}}$

ii) $x_n = \dfrac{n}{\sqrt[n]{n!}}$

iii) $x_n = n^{\frac{1}{n}}$

Can you evaluate any these more directly?

Q7) **Tail of a sequence:** It's been mentioned several times that changing the first few terms of a sequence does not change its convergence, or the limit. Let's make this precise: Suppose $x_n \to L$ as $n \to \infty$. Set $y_1 = A_1, y_2 = A_2, \ldots, y_{M-1} = A_{M-1}$ and $y_n = x_n$ for $n \geq M$. Give an ϵ-N proof that $y_n \to L$ too.

Q8) It's also been mentioned that 'shifting' a sequence does not change its convergence, or the limit, either. Let's make this precise: Suppose $x_n \to L$ as $n \to \infty$. Set $y_n = x_{n+k}$. Give an ϵ-N proof that $y_n \to L$ too.

4 Completeness

Q1) Recall the completeness of $\mathbb{R}$: every non-empty subset $S \subset \mathbb{R}$ which is bounded above has a supremum $\sup S \in \mathbb{R}$. Show that this does not hold if $\mathbb{R}$ is replaced by $\mathbb{Q}$ by looking at the following example:

$$S = \{x \in \mathbb{Q} \mid x^2 < 2\}$$

Specifically, show that no $q \in \mathbb{Q}$ can be the least upper bound of S. So we can conclude $\mathbb{Q}$ is not complete.

$^{\#}$Q2) **Cauchy Criterion:** A sequence x_n is called *Cauchy* if: given any $\epsilon > 0$, there exists N such that for $n, m \geq N$, we have $|x_n - x_m| < \epsilon$. (Eventually all terms are within ϵ of each other.) Working in $\mathbb{R}$, prove the following are equivalent:

a) The sequence x_n converges (to $L \in \mathbb{R}$).

b) The sequence x_n is Cauchy.

Hint: For a) $\implies$ b), write $x_n - x_m$ as $(x_n - L) + (L - x_m)$, and use the ϵ-N definition of $x_n \to L$. (With what choice of ϵ?)

For b) $\implies$ a), first show $\{x_n\}$ is bounded. Then use Bolzano-Weierstrass to find a convergent subsequence $x_{n_i} \to L$. And finally show $x_n \to L$, itself.

Note: a) $\implies$ b) always holds, but showing b) $\implies$ a) requires using the completeness of $\mathbb{R}$. That every Cauchy sequence converges can be taken as a *definition* of completeness. You might meet the concept of a Cauchy sequence again Complex Analysis 2.

5 Infinite Series Convergence Tests

Q1) **Root Test:** Let $\sum_{n=1}^{\infty} a_n$ be a series, and suppose that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists. Prove that:

- if $L < 1$ then the series converges absolutely,
- if $L > 1$ then the series diverges.

Can you say anything if $L = 1$? What if you also know that $\sqrt[n]{|a_n|} \to 1$ from above, i.e. eventually all $\sqrt[n]{|a_n|} \geq 1$?

Hint: Imitate the proof of the ratio test.

$\Rightarrow$Q2) Use the root test above to determine whether or not the following series converge:

i) $$\sum_{n=2}^{\infty} \frac{1}{\log(n)^n}$$

ii) $$\sum_{n=2}^{\infty} \left[\log\left(\frac{3n^2+n}{n^2-2n+1}\right)\right]^n$$

iii) $$\sum_{n=3}^{\infty} \left[\log\left(\frac{2n^3-5n^2}{n^3+4n-3}\right)\right]^n$$

iv) $$\sum_{n=1}^{\infty} \left[3n^6 \sin^3\left(\frac{2n+7}{3n^3+5n^2-6n-1}\right)\right]^n$$

Q3) **Generalised Comparison:** Suppose a_n, b_n, c_n are three sequences such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{Z}_{>0}$. Prove that:

- If the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} c_n$ both converge, then $\sum_{n=1}^{\infty} b_n$ also converges.
- If the series $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$, then $\sum_{n=1}^{\infty} b_n$ also diverges to $+\infty$.
- If the series $\sum_{n=1}^{\infty} c_n$ diverges to $-\infty$, then $\sum_{n=1}^{\infty} b_n$ also diverges to $-\infty$.

Hint: For the first part, what can you do with $a_n \leq b_n \leq c_n$ to make the normal comparison test applicable? For the second and third, partial sums?

Note: Since you haven't seen this in lectures, you may *not* use this when solving homework or exam questions *unless* you prove it first. If you *really* feel the need to use this generalisation, it may be worth looking at $\sum_{n=1}^{\infty} |x_n|$ instead.

Q4) Recall the Alternating Sign Test from lectures: It says that if the sequence y_n is positive, decreasing, and $\lim_{n\to\infty} y_n = 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^n y_n$ converges.

i) Firstly, show that the condition that y_n is positive can be removed: If y_n is a decreasing sequence, and $\lim_{n\to\infty} y_n = 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^n y_n$ converges.

Hint: Use y_n is decreasing, and $\lim_{n\to\infty} y_n = 0$ to show $y_n \geq 0$ for all n. What happens if $y_N = 0$ for some N, and otherwise?

*ii) Now show that the other two hypotheses are necessary, even if you restrict to y_n positive.

That is, find a sequence a_n which is positive, decreasing, but doesn't have limit 0, such that the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ doesn't converge.

And find another sequence b_n which is positive, has limit 0, but is not decreasing, such that the alternating series $\sum_{n=1}^{\infty}(-1)^n b_n$ also doesn't converge.

Note: This is a nice example of how mathematicians think. When told of a theorem requiring many hypotheses, a mathematician will naturally wonder whether all the hypotheses are necessary. Where are these conditions used in the proof? Does theorem fail if I weaken or remove any of the hypotheses? Can I generalise the theorem in any way?

*Q5) **Cauchy Condensation Test:** Suppose a_n is a positive, decreasing sequence. By showing:

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} 2^n a_{2^n} \leq 2\sum_{n=1}^{\infty} a_n \,,$$

conclude that:

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} 2^n a_{2^n} \text{ converges}$$

Hint: Write out the partial sums.

Note: This generalises the proof from lectures that the harmonic series diverges. Taking $a_n = \frac{1}{n}$, one gets $\sum_{n=1}^{\infty} \frac{1}{n}$ converges if and only if $\sum_{n=1}^{\infty} 2^n \frac{1}{2^n} = \sum_{n=1}^{\infty} 1$ converges, and the latter series obviously diverges.

You will *not* need to use this in any homework, tutorial of exam questions. Just consider it a glimpse of the vast zoo of more specialised convergence tests.

⇒Q6) In lectures you used the integral test to show the following series diverges. This time, use Cauchy's Condensation Test to show it diverges:

$$\sum_{n=2}^{\infty} \frac{1}{n\,\log(n)}$$

6 Infinite Series Convergence

Q1) True or false?

i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ converges since $1 + \frac{1}{n} > 1$.

ii) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n^2}\right)^{n^2}$ converges.

Q2) Do the following series converge?

i) $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{n^{n-1}} \left(\frac{19}{7}\right)^{n-1}$

Hint: You may assume that $\frac{19}{7} < e$.

ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\log(n!)}$

Hint: How do n^n and $n!$ compare?

iii) $\displaystyle\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$

Hint: What are the partial sums?

Q3) Use the integral test to determine whether the following series converge:

i) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\,\log(n)\,\log(\log(n))}$

ii) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\,\log(n)^2}$

iii) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\,\sqrt{\log(n)}}$

iv) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\,\log(n)\,\log(\log(n))^{1/3}}$

v) $\displaystyle\sum_{n=15}^{\infty} \frac{1}{n\,\log(n)\,\log(\log(n))\,\log(\log(\log(n)))}$

Considering the above examples, can you guess a more general result? Can you prove it?

Note: Series like the above, and things involving nested logarithms appear a lot in analytic number theory. This leads to the appalling joke: What sound does a drowning analytic number theorist make? Log, log, log, ...

Q4) Linking to analytic number theory... If p_n is the n-th prime number, Dusart's inequality says, for $n \geq 6$:

$$p_n < n\log(n) + n\log(\log(n))\,.$$

Use this to prove the divergence of the *prime harmonic series*:

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

Note: Isn't this a little bit impressive? Even after getting rid of all the composite numbers from the harmonic series, the sum *still* diverges. So in some sense you might say there are more primes than squares, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Q5) Find all values of p for which the following series converges:

$$\sum_{n=2}^{\infty} \frac{1}{n\ \log(n)^p}$$

Q6) You can determine whether the following series converge or diverge using other means, but for this question use the integral test to determine whether they converge or diverge. Make sure the function satisfies all the hypotheses of the integral test!

i) $\displaystyle\sum_{n=1}^{\infty} n^2 \exp(-n)$

ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\log(n)}{n^2}$

iii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

iv) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{x}}$

7 Infinite Series Theorems

Q1) **Tail of a series:** It's been mentioned that only the tail of an infinite series is important when discussing convergence. Make this precise: By looking at the partial sums, show that $\sum_{n=1}^{\infty} x_n$ converges if and only $\sum_{n=M}^{\infty} x_n$ converges.

#Q2) **Riemann Series Theorem:** Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series.

i) Show that a conditionally convergent series must have an infinite number of positive terms and an infinite number of negative terms.

ii) Take:

$$a_n^+ := \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a_n^- := \begin{cases} 0 & \text{if } a_n \geq 0 \\ a_n & \text{otherwise} \end{cases},$$

so that a_n^+ keeps the positive terms and replaces the negative terms with 0s, while a_n^- keeps the negative terms and replaces the positive terms with 0s. Observe that: $\sum a_n = \sum a_n^+ + \sum a_n^-$. Use this to conclude that $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ diverge to $+\infty$ and $-\infty$ respectively.

*iii) Pick any $M \in \mathbb{R}$. Use the previous part to show that you can rearrange the terms of the sequence a_n to get a new sequence b_n whose sum $\sum_{n=1}^{\infty} b_n = M$.

Hint: For any $r > 0 \in \mathbb{R}$, since $\sum a_n^+$ diverges to $+\infty$, the partial sums are eventually greater r. Once you have added enough positive terms to exceed M, make use of the negative terms. (Convince yourself that you've used every term a_n once and only once.)

*iv) Can you do the same but get that $\sum_{n=1}^{\infty} b_n$ diverges to $+\infty$, or diverges to $-\infty$, or diverges by oscillation[1]

Note: This question shows how dangerously, badly wrong some naïve ways of manipulating infinite series can go: you can't rearrange a convergent series and be sure you've still got convergent series, let alone one with the same sum, at least for general infinite series. However, it can be shown that when you rearrange an absolutely convergent series, the rearranged series always converges and it converges to the *same* sum as the original series. (Absolutely convergent series are very well behaved.)

8 Power Series

Q1) **Dilogarithm and polylogarithms:** Recall the Taylor series for $-\log(1-x)$ is given by

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n},$$

and this converges for $|x| < 1$. By replacing n with n^2 we get the definition of the *dilogarithm* function

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

More generally the *p-th polylogarithm* (for $p = 1, 2, 3, \ldots$), which is defined by

$$\mathrm{Li}_p(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}.$$

i) What is the radius of convergence of the power series defining $\mathrm{Li}_p(x)$?

ii) Using the fact that you can differentiate a power series term-by-term, find $\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{Li}_p(x)$ (in terms of Li_{p-1}).

*iii) Check the following identity for the dilogarithm

$$\mathrm{Li}_2(x^2) = 2(\mathrm{Li}_2(x) + \mathrm{Li}_2(-x))$$

(Try adding the power series together.)

[1] That is, the series diverges but the partial sums don't go to $\pm\infty$, the partial sums remain bounded.

*iv) A similar identity holds for every polylogarithm. What should the coefficient λ be to make the following identity true?

$$\mathrm{Li}_p(x^2) = \lambda(\mathrm{Li}_p(x) + \mathrm{Li}_p(-x))$$

Note: The dilogarithm and higher polylogarithm functions are of considerable interest to both number theorists and particle physicists.

One of the main focuses is finding 'functional equations', identities like the two examples above. For example, the fundamental property of the (usual) logarithm is that it turns multiplication into addition

$$\log(xy) = \log(x) + \log(y)\,.$$

(You can write this in terms of $\mathrm{Li}_1(x) = -\log(1-x)$ if you want.)

Similarly, the dilogarithm satisfies the so-called '5-term' equation

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2\left(\tfrac{1-x}{1-xy}\right) + \mathrm{Li}_2(1-xy) + \mathrm{Li}_2\left(\tfrac{1-y}{1-xy}\right) =$$
$$\tfrac{\pi^2}{6} - \log(x)\log(1-x) - \log(y)\log(1-y) + \log\left(\tfrac{1-x}{1-xy}\right)\log\left(\tfrac{1-y}{1-xy}\right)\,.$$

This equation is the fundamental property of the dilogarithm. (Where does $\frac{\pi^2}{6}$ come from? Try looking up the Basel Problem.)

Physicists are interested in these functional equations because of how drastically they can simplify answers to certain computations. An answer which used to span 14 pages can be condensed to just 4 lines with good knowledge of the properties of polylogarithms!

See `http://en.wikipedia.org/wiki/Dilogarithm` and `http://en.wikipedia.org/wiki/Polylogarithm` for much more information.

9 Riemann Integrals

Q1) Let $\mathcal{P}_n([a,b])$ be the partition of the interval $[a,b]$ into n equal parts. Write down the upper and lower Riemann sums for the given function, on the given partition of the given interval, and numerically evaluate the results:

i) $f(x) = x$ for $\mathcal{P}_5([0,3])$

ii) $f(x) = x$ for $\mathcal{P}_{10}([0,3])$

iii) $f(x) = x^2$ for $\mathcal{P}_4([0,1])$

iv) $f(x) = x^2$ for $\mathcal{P}_5([0,1])$

v) $f(x) = x^2$ for $\mathcal{P}_6([0,1])$

vi) $f(x) = \sqrt{x}$ for $\mathcal{P}_4([0,4])$

vii) $f(x) = \sqrt{x}$ for $\mathcal{P}_5([0,4])$

viii) $f(x) = \sqrt{x}$ for $\mathcal{P}_6([0,4])$

ix) $f(x) = e^x$ for $\mathcal{P}_5([1,2])$

x) $f(x) = e^x$ for $\mathcal{P}_6([1,2])$

xi) $f(x) = \log x$ for $\mathcal{P}_5([1,2])$

xii) $f(x) = \log x$ for $\mathcal{P}_5([1,2])$

xiii) $f(x) = \sin x$ for $\mathcal{P}_5([0,\pi])$

xiv) $f(x) = \sin x$ for $\mathcal{P}_5([0,\pi])$

Q2) Write down the upper and lower Riemann sums for the given function on the partition $\mathcal{P}_n([a,b])$ of the interval $[a,b]$ into n equal parts. Use this to show that the function is Riemann integrable on the given interval

i) $f(x) = x$ on $[0,1]$

ii) $f(x) = x$ on $[0,2]$

iii) $f(x) = x$ on $[0,a$

iv) $f(x) = x^2$ on $[0,1]$

v) $f(x) = x^2$ on $[0,2]$

vi) $f(x) = x^2$ on $[0,a]$

vii) $f(x) = \sin(x)$ on $[0,\pi]$ (For ease, do this for $\mathcal{P}_{2n+1}([0,\pi])$.)

*Q3) By viewing the following as a Riemann sum for a particular function on some interval, express the following limits as a definite integral, and evaluate the integral to find the limit:

i) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\sqrt{\frac{i}{n}}$

ii) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\sqrt{\frac{i-1}{n}}$

iii) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(1+\frac{i}{n}\right)^2$

iv) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(1+\left(\frac{i}{n}\right)^2\right)$

v) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(1+\left(1+\frac{i}{n}\right)^3\right)$

*vi) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2+i^2}$

10 Uniform Convergence

Q1) Let $f_n(x) = \frac{1}{x^n}$, for $x \in [1,\infty)$, and $n = 1,2,3,\ldots$. Show that f_n converges (pointwise). Show that f_n converges uniformly on $[R,\infty)$, for $R > 1$. Does it converge uniformly on $[1,\infty)$?

Q2) Find the pointwise limit of the sequence of functions $f_n = \frac{nx^2+1}{nx+1}$, on the interval $[1,2]$. Show that the convergence is uniform on this interval.

Q3) Fix $0 < M < 1 \in \mathbb{R}$. Show that

$$\sum_{n=0}^{\infty} x^n$$

converges uniformly for $|x| < M$. Does it converge uniformly for $|x| < 1$?

Q4) Fix $M > 0 \in \mathbb{R}$. Show that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges uniformly for $|x| < M$. Does it converge uniformly for all $z \in \mathbb{R}$?

*Q5) Recall the ϵ-N definition of uniform convergence of functions f_n to f on some region I:

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{Z}_{>0} \text{ such that } \forall x \in I \text{ we have}$$
$$n > N \implies |f_n(x) - f(x)| < \epsilon$$

Use this to show that if each f_n is continuous on I, the limit f is also continuous on I.

Hint: Make use of $f(z) - f(a) = (f(z) - f_n(z)) + (f_n(z) - f_n(a)) + (f_n(a) - f(a))$. Can you make each bit $< \epsilon/3$?

11 Improper Integrals

Q1) Show that the following integral converges absolutely:

$$\int_0^{\infty} \frac{\cos mx}{1 + x^2} \, \mathrm{d}x$$

Note: In Complex Analysis 2 you will learn how to exactly evaluate this integral. Its exact value is $\frac{1}{2}\pi e^{-m}$. How does this compare with the upper bound you use above?

*Q2) **Fresnel Integral:** By using the change of variables $t = x^2$ on $\int_0^s \sin(x^2) \, \mathrm{d}x$, and integrating the result by parts, show that:

$$\int_0^{\infty} \sin(x^2) \, \mathrm{d}x$$

converges.

Note: This shows that, unlike for infinite series, $f(x) \to 0$ as $x \to \infty$ is not necessary for $\int_0^{\infty} f(x) \, \mathrm{d}x$ to converge, since $\lim_{x\to\infty} \sin(x^2)$ does not exist. Although as Question 75 on the main problem sheet shows: if $\lim_{x\to\infty} f(x)$ exists, then it must be 0.