Analysis Collection 2013
Brief Solutions

Question 1

Compute $\lim_{n\to\infty} x_n$ (if it exists), where $x_n = [n^4 e^{-n} + n^3 + n^2 \sin(n)]/\sqrt{1 + 25n^6 + \log(n)}$.

Solution: Divide top and bottom by n^3, then you can see $\lim_{n\to\infty} x_n = \frac{1}{5}$

Question 2

Compute $\lim_{n\to\infty} x_n$, where $x_n = \{1 - 1/(3n+1)\}^n$, or show that there is no limit.

Solution: Put x_n into a form where you can use $(1 + c/n)^n \to e^c$ as $n \to \infty$. It follows that $\lim_{n\to\infty} x_n = e^{-1/3}$.

Question 3

If $x_n < 1$ for all n, and $x_n \to L$ as $n \to \infty$, prove that $L \leq 1$.

Solution: Set $b = 1$ in the solution to Question 23 on the Problem Sheet.

Question 4

Consider $\{x_n = \frac{n+2}{n+1}\}$. Give an ($\epsilon$-$N$) proof that $\lim_{n\to\infty} x_n = 1$.

Solution:

$$|x_n - 1| = \left|\frac{1}{n+1}\right|$$

Given $\epsilon > 0$, take $N = \frac{1}{\epsilon} - 1$ (or just $= \frac{1}{\epsilon}$, for simplicity...). Then $n > N$ implies $|x_n - 1| < \epsilon$.

Question 5

Find the supremum and the infimum of the function $f(x) = x^2/(x^2 + 5)$ for $x > 0$.

Solution: Infimum: $f(x) \geq 0$, and $f(x) \to 0$ as $x \to 0$, so $\inf(f) = 0$.
Supremum: $f(x) \leq 1$, and $f(x) \to 1$ as $x \to \infty$, so $\sup(f) = 1$.

Question 6

Discuss the convergence of the series $\sum_{n=1}^{\infty} x_n$, where $x_n = \sin((n + \frac{1}{2})\pi)\frac{\log(n)}{n+2}$.

Solution: Since $\sin((n+1/2)\pi) = (-1)^n$, use the alternating series test. Check that $y_n = \log(n)/(n+2)$ is positive, decreasing (eventually), and tends to 0. Hence the series converges. (To see y_n is decreasing eventually, look at:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\log(x)}{x+2} = \frac{x+2-x\log(x)}{x(x+2)^2}$$

This is eventually < 0 since $x+2-x\log(x) \to -\infty$ as $x \to \infty$.)

Question 7

For what values of α does the series $\sum_{n=1}^{\infty} x_n$ converge, where $x_n = n^{\alpha}\left\{\frac{1}{n+1} - \frac{1}{\sqrt{n^2+n}}\right\}$.

Solution: $n^{2-\alpha}x_n \to -\frac{1}{2}$ as $n \to \infty$, so eventually $-0.51n^{\alpha-2} \leq x_n \leq -0.49n^{\alpha-2}$ (or any other bounds which work...). So the series converges iff $\alpha - 2 < -1$ iff $\alpha < 1$.

Question 8

Find values of z for which the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges when $z_0 = 1-i$, and $a_n = (1+i)^n$.

Solution: This is a geometric series with common ratio $r = (1+i)(z-(1-i))$, so converges exactly when $|z-(1-i)| < \frac{1}{|1+i|} = \frac{1}{\sqrt{2}}$.