Series Convergence/Divergence Tests

Also see `http://www.maths.dur.ac.uk/users/steven.charlton/analysis1_1314`

Below is a list of the main series convergence test seen during the Analysis 1 lecture course. A few more specialised ones from the online lectures notes have been skipped (Raabe's test and Gauss's test).

Divergence/n-th Term: If $\lim_{n\to\infty} a_n \neq 0$, including the case that the limit does not exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

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p-Series: For p constant, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff p > 1$.

Geometric Series: For r constant, the series $\sum_{n=1}^{\infty} r^n$ converges $\iff |r| < 1$.

Absolute Convergence: If $\sum_{n=1}^{\infty} |x_n|$ converges, then $\sum_{n=1}^{\infty} x_n$ also converges. That is, if the series $\sum_{n=1}^{\infty} x_n$ converges absolutely, then it also (just plain old) converges.

Alternating Sign Test: Suppose y_n is a positive, decreasing sequence which has limit $\lim_{n\to\infty} y_n = 0$. Then $\sum_{n=1}^{\infty} (-1)^n y_n$ converges

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Comparison Test: Suppose $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$, then:

- If $\sum_{n=1}^{\infty} y_n$ converges, then $\sum_{n=1}^{\infty} x_n$ also converges.
- If $\sum_{n=1}^{\infty} x_n$ diverges, then $\sum_{n=1}^{\infty} y_n$ also diverges.

Limit Comparison/Quotient Test: Suppose $x_n \geq 0$ and $y_n > 0$ for all $n \in \mathbb{N}$, and that $\lim_{n\to\infty} \frac{x_n}{y_n} = L$, with $0 < L < \infty$. Then the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ either both converge, or both diverge.

[You reduce this to the usual comparison test by saying eventually $L - \epsilon < \frac{x_n}{y_n} < L + \epsilon$, taking ϵ small enough that $L - \epsilon > 0$, and multiplying through by y_n.

Knowing this you can then also say something in the case $L = 0$: eventually $\frac{x_n}{y_n} \leq 1$, so eventually $x_n \leq y_n$. Combine this with the given $x_n \geq 0$ to get eventually $0 \leq x_n \leq y_n$, then what does comparison say?]

Ratio Test: Suppose x_n is a sequence of non-zero numbers such that $\lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| = L$. Then

- If $L < 1$, the series $\sum_{n=1}^{\infty} x_n$ converges absolutely.
- If $L > 1$, the series $\sum_{n=1}^{\infty} x_n$ diverges.
- If $L = 1$, inconclusive. Try something else.

Root Test: Suppose x_n is a sequence of numbers such that $\lim_{n\to\infty} \sqrt[n]{|x_n|} = L$. Then

- If $L < 1$, the series $\sum_{n=1}^{\infty} x_n$ converges absolutely.
- If $L > 1$, the series $\sum_{n=1}^{\infty} x_n$ diverges.
- If $L = 1$, inconclusive. Try something else.

[If you forget whether $L < 1$ means convergence or divergence in either of the above, just apply the test to geometric series $\sum_{n=1}^{\infty} r^n$ and remember a geometric series converges $\iff |r| < 1$.]

Integral Test: Suppose $f(x)$ is a positive decreasing function on $[1, \infty)$. Set $x_n = f(n)$, for $n \in \mathbb{N}$, and $F(m) = \int_1^m f(t)\,\mathrm{d}t$, for $m \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} x_n$ converges $\iff F(m)$ has a limit as $m \to \infty$.

[Draw a series of rectangles of height $x_n = f(n)$ and width 1 below the graph $y = f(x)$ and another series of rectangles (offsetting them by 1) above the graph $y = f(x)$. Up to one term, the sum gives upper and lower bounds on the integral, so the sum converges iff the integral converges.]

Series Convergence/Divergence Strategy

Here are a tips, tricks and strategies for deciding whether the series $\sum_{n=1}^{\infty} x_n$ converges, and for deciding which convergence test to try on a particular series.

This list is not exhaustive. There is no universal strategy which will always tell you whether some arbitrary series converges or diverges – there are plenty of series whose convergence or divergence we can't yet determine – but this should be enough to handle the majority series which turn up in the Analysis 1 course.

Go through a lot of practice questions, get a feeling for how things work. You'll develop an instinct for which tests should work for a given series.

If it does fail, if you stumble upon a series which you can't handle, maybe you made a mistake in or gave up too quickly with a particular convergence test? Double-check your working, and persevere with any test you try. Otherwise, you may have to get creative.

- Don't overlook the obvious. Do the terms x_n tend to 0? Take a moment to check they do, so you don't waste time on a series which obviously diverges by the **divergence/n-th term test**.
- Do you recognise the series? Is it a **p-series**, or a **geometric series**? If so use the convergence results you know about them. Can you express it as a linear combination of known convergent series? If so use the **linearity** of series.
- Are the terms of the series (eventually) positive? If not, do the signs alternate: is there a $(-1)^n$ around, it may be disguised as $\cos(n\pi)$ or $\sin(n\pi + \frac{\pi}{2})$? If the signs alternate, try the **alternating sign test**. Otherwise take absolute values $|x_n|$, and try to show the series $\sum_{n=1}^{\infty} x_n$ **converges absolutely**. Remember the triangle inequality $|x + y| \leq |x| + |y|$.
- For ratios of polynomials (including k-th roots), use the heuristic approximations to see what the series behaves like. Then try **limit comparison/quotient** with this approximation to make it precise. Maybe you can also get to some multiple of the approximation by using inequalities, and then use the **comparison test**.
- If the series involves trig functions, you can probably use the bounds $|\sin(x)| \leq 1$, and $|\cos(x)| \leq 1$, to **compare** with a simpler series. If the series has exponentials, maybe think of them as very large degree polynomials; and logrithms, very small (fractional) degree polynomials. Use exponentials beat powers, and powers beat logs to make this precise.
- If x_n involves factorials $n!$, definitely try the **ratio test**. The **ratio test** can also work for terms involving powers like $2^n, n^n$ since there will be a lot of cancellation, or simplication in $|x_{n+1}/x_n|$.
- If x_n is of the form ${y_n}^n$, try the **root test**, since $\sqrt[n]{|x_n|} = y_n$ is a simpler sequence.
- If you are unsure of what to try, it can't hurt to quickly try the **ratio or root tests** since you don't have to make any clever choices when applying them.
- If $x_n = f(n)$, and $f(x)$ is a function that you can 'easily' integrate, try the **integral test**. For quite a number of series the integral test will work, but might be overkill. Things like rational functions would easily fall to (limit) comparison instead. If you stumble across something like $1/(n \log(n) \log(\log(n)))$, you do need the **integral test**.

If you have to find the radius of convergence of a power series, use the **ratio test** (or maybe the **root test**).