Analysis Tutorial Week 5

Solutions

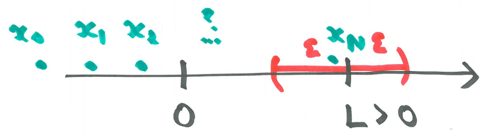
Question 17

Question: If $\{x_n\}$ is a sequence such that $x_n \to L$, and $x_n < 0$ for all n, prove that $L \leq 0$. Is it necessarily true that $L < 0$?

Sketch: We want to try to show this by contradiction.

Recall that $x_n \to L$ means that for any $\epsilon > 0$ we choose, we can find N such that if $n > N$, then $|x_n - L| < \epsilon$. "After some point all the terms of the sequence x_n are within ϵ of the limit L."

Heuristically we want to argue that if $L > 0$, eventually the terms x_n are so close to L that they must be positive, which will contradict the hypothesis $x_n < 0$ for all n, given in the question. A picture might be illuminating:



But we need to do this formally. First some sketchy workings...

Suppose that the limit L of the sequence is > 0. Then for $\epsilon > 0$, we can find N such that $n > N$ implies $|x_n - L| < \epsilon$. Let's expand this to see what choice of ϵ might be good:

$$\begin{aligned} |x_n - L| < \epsilon &\iff -\epsilon < x_n - L < \epsilon \\ &\iff L - \epsilon < x_n < \epsilon + L \end{aligned}$$

Can we choose $\epsilon > 0$ so that $L - \epsilon > 0$? Yes, the picture might suggest to try something like $\epsilon = L/2$, half the distance from 0 to L. Then $L - \epsilon = L - L/2 = L/2 > 0$. Now we have the pieces we need to write out the formal proof.

Can we improve $L \leq 0$ to $L < 0$? Probably not, but we'd need to find a counterexample to prove this. One of the first sequences you've seen is $x_n = 1/n$ which has limit 0, even though all the terms are positive. Can we tweak this to make it work?

Solution: Suppose that the limit L of the sequence x_n is > 0. If we take $\epsilon = L/2 > 0$, then by the ϵ-n definition of a limit we can find N_0 such that $n > N_0$ implies $|x_n - L| < \epsilon = L/2$. But then:

$$\begin{aligned} |x_n - L| < L/2 &\implies -L/2 < x_n - L < L/2 \\ &\implies L - L/2 < x_n < L + L/2 \\ &\implies L/2 < x_n < 3L/2 \end{aligned}$$

In particular $x_n > L/2 > 0$.

So we have $n > N_0$ implies $x_n > 0$, which contradicts the hypothesis of the question that $x_n < 0$ for all n. Hence our assumption was wrong and so $L \leq 0$.

It is not necessarily true that $L < 0$. Consider $x_n = -1/n$. Certainly $x_n < 0$ for all n, but we have that $x_n \to 0$.

Question 25

Question: Prove that one of the following statements is true and that the other is false.

(a) If $x_n \to 1$ as $n \to \infty$, then $(x_n)^n \to 1$ as $n \to \infty$.

(b) If $0 < r < 1$ and $x_n \to r$ as $n \to \infty$, then $(x_n)^n \to 0$ as $n \to \infty$.

Sketch: After doing some of the earlier tutorial questions, and the homework, the result that $(1 + \frac{c}{n})^n \to e^c$ should be fresh in your mind. This should be fast becoming a favourite (counter)example.... So (a) is false. Now we have to prove (b) is true.

Heuristically, if $x_n \to r$, then eventually x_n is very close to r, it is eventually within some small ϵ of r. Since $0 < r < 1$, this means that eventually $0 < r - \epsilon < x_n < r + \epsilon < 1$, so we can squeeze x_n to 0. (Try drawing a picture like above...)

How to make this rigorous? We'll want to choose ϵ so that $r + \epsilon < 1$. Like above, let's choose ϵ to be half the distance from r to 1. So $\epsilon = \frac{1-r}{2}$. Then $r + \epsilon = \frac{1+r}{2} < 2/2 = 1$. Do we have to worry about the lower bound not being good enough? No, since $r > 0$, taking away such a small epsilon won't get us down far enough to be problematic. And anyway we are going to consider $|x_n^n|$ rather than x_n^n. So we're good to go.

Solution:

(a) is false: Consider $x_n = 1 + \frac{1}{n}$. Certainly $x_n \to 1 + 0 = 0$, by COLT. But $x_n^n = (1 + \frac{1}{n})^n \to e$ is a result from lectures, and $e \neq 1$.

Another very nice counter-example is something like $x_n = 7^{1/n}$. You can check easily that $x_n \to 7^0 = 1$, but obviously $x_n^n = 7 \to 7$, and definitely $7 \neq 1$.

(b) is true: Take $\epsilon = \frac{1-r}{2} > 0$, then we can find N_0 such that $n > N_0$ implies $|x_n - r| < \epsilon = \frac{1-r}{2}$. Then:

$$|x_n| = |x_n - r + r| \leq |x_n - r| + |r|$$

using the triangle inequality

$$= |x_n - r| + r$$
$$< \tfrac{1-r}{2} + r = \tfrac{1+r}{2}$$

So we get:

$$|x_n^n| = |x_n|^n < (\tfrac{1+r}{2})^n$$

Since $0 < \frac{1+r}{2} < 1$ is *constant*, the sequence $(\frac{1+r}{2})^n \to 0$. Hence $x_n^n \to 0$ as well, by squeezing.

(It doesn't matter what happens with the first N_0 terms of the sequence, since this is only a fixed finite number of terms – the 'tail' of the sequence is all that matters.)