Problem sheet - Multiple Zeta Values
Starred* questions are bonus questions.

Discussion on 09.07 at 11:30: Ex 10.1 – Ex 10.4
Discussion on 02.07 at 11:30: Ex 9.1 – Ex 9.3

Keywords for the week 29.06.20–05.07.20: (Drinfeld) associator, Mould, Bimould, Alternal, Alternil

Exercise 10.1: (5 points)
Let $a = a_{i_1} a_{i_2} \cdots a_{i_l}$ and define the mould $M^\bullet \colon A \to k(u_1, u_2, u_3, \ldots)$ by

$$(M \mid a) = \begin{cases} 0 & \text{if } a = \emptyset \text{ or if } a \text{ has a repeated letter, and} \\ \frac{1}{u_{i_2} - u_{i_1}} \frac{1}{u_{i_3} - u_{i_2}} \cdots \frac{1}{u_{i_\ell} - u_{i_{\ell-1}}} & \text{otherwise}. \end{cases}$$

a) Check that for depth $\ell \leq 4$, the condition for M to be alternal, i.e.

$$M(u_1, \ldots, u_r) \sqcup M_(u_{r+1}, \ldots, u_{r+s}) = 0$$

for $r, s \geq 1$, $r + s \leq 4$ is satisfied.

b*) Write down all the equations in depth $\ell = 5$ which must be satisfied for M to be alteral. Check these equations hold.

Exercise 10.2: (5 points)
a) Let $M^\bullet \colon A^* \to k[\![u_1, v_1, u_2, v_2, u_3, v_3, \ldots]\!]$ be a bimould, and define the *swap* by

$$\operatorname{swap}\left(M \begin{pmatrix} u_1, & u_2, & \ldots, & u_\ell \\ v_1, & v_2, & \ldots, & v_\ell \end{pmatrix} \right) = M \begin{pmatrix} v_\ell, & v_{\ell-1} - v_\ell, & \ldots, & v_1 - v_2 \\ u_1 + \cdots + u_\ell, & u_1 + \cdots + u_{\ell-1} & \ldots, & u_1 \end{pmatrix}.$$

Check that the function composition $\operatorname{swap} \circ \operatorname{swap} = \operatorname{id}$ holds.

b) Let $m = (0, \ldots, 0, m_r, m_{r+1}, \ldots,)$ be an alternil bimould such that $m_i = 0$ for $i \leq r$ and $m_r \neq 0$. Show that the mould $(0, \ldots, 0, m_r, 0, \ldots)$ concentrated in depth r (by abuse of notation just m_r), is an alternal bimould.

Exercise 10.3: (5 points)
let $F_\ell(x_1, \ldots, x_\ell) = \sum_{s_1, \ldots, s_\ell \geq 1} \zeta^*(s_1, \ldots, s_l) x_1^{s_1 - 1} \cdots x_\ell^{s_\ell - 1}$ be the generating series of stuffle regularised multiple zeta values of depth ℓ. Viewing $F = (F_0, F_1(x_1), F_2(x_1, x_2), \ldots)$ as a mould show that

$$F_r(x_1, \ldots, x_r) * F_s(x_{r+1}, \ldots, x_{r+s}) = F_r(x_1, \ldots, x_r) \cdot F_s(x_{r+1}, \ldots, x_{r+s}).$$

Exercise 10.4*: (5 points)

Let

$$\Phi = \sum_{w \in \{x_0, x_1\}^*} \zeta^{ш}(w) w$$

be the Drinfeld associator. Check that Φ is group-like, i.e. under the coproduct given by $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$ (so x_i is primitive), we have $\Delta\Phi = \Phi \widehat{\otimes} \Phi$.

Hint: Make use of the criterion of Friedrichs (see Satz 3.8 in the Masters Thesis of A. Burmester).

Keywords for the week 22.06.20–28.06.20: Broadhurst-Kreimer conjecture, Hilbert-Poincaré series, Hopf algebra, Lie algebra

Exercise 9.1: (5 points)

a) Part of Zagier's conjecture claims that the algebra $\mathcal{Z}$ of MZV's is graded by the weight. Assuming this, show that $\mathcal{Z}$ is a filtered, graded algebra (filtered by the depth, and graded by the weight).

b) Let k be a field, and let A and B be filtered graded k-algebras. Show that $A \otimes_k B$ is again a filtered graded k-algebra.

Hint: (See also, Deligne "Théorie de Hodge".)

c*) The show that the map $(\mathbb{Q}\langle t_2, t_3 \rangle, ⧢) \to \mathcal{Z}$, $t_{i_1} \cdots t_{i_k} \mapsto \zeta(i_1, \ldots, i_k)$ cannot extend to a homomorphism.

Exercise 9.2: (5 points)

Recall

$$\mathrm{BK}^0(x,y) = \frac{1}{1 - O_3(x)y + S(x)y^2 - S(x)y^4}\,,$$

where

$$O_3(x) = \frac{x^3}{1-x^2} = x^3 + x^5 + x^7 + \cdots$$

$$S(x) = \frac{x^{12}}{(1-x^4)(1-x^6)} = x^{12} + x^{16} + x^{18} + \cdots\,.$$

Define $(g_{k,\ell})_{k\geq 3, \ell \geq 1}$ through

$$\prod_{k\geq 3, \ell\geq 1} (1 - x^k y^\ell)^{-g_{k,\ell}} = \mathrm{BK}^0(x,y)\,.$$

Assuming the Broadhurst-Kreimer conjecture, then $g_{k,\ell}$ is the (conjectural) number of generators of $\mathrm{gr}^F(\mathcal{Z}^0)$ in weight k and depth ℓ.

a*) Using the Möebius inversion formula (Ex 8.1 below), explain why

$$g_{k,\ell} = \sum_{d|(k,\ell)} \frac{\mu(d)}{d} b_{k/d,\ell/d}\,,$$

where $b_{k,\ell}$ is the coefficient of $x^k y^\ell$ in $\log \mathrm{BK}^0(x,y)$.

b) Show that

$$O_3(x)^3 = \sum_{\substack{k\geq 9\\ \text{odd}}} \frac{(k-5)(k-7)}{8} x^k\,.$$

Check also that

$$S(x)O_3(x) = x^{15}\Big(\frac{11 + 6x^2 - 9x^4}{48(1-x^2)^3} + \frac{1}{8(1+x^2)} + \frac{31 + 15x^2 + 15x^4}{48(1-x^6)}\Big)\,,$$

hence give a formula for the coefficient of x^k in $S(x)O_3(x)$.

c) Prove that for k odd, we have

$$g_{k,3} = \left\lfloor \frac{(k-3)^2-1}{48} \right\rfloor .$$

Exercise 9.3***:** (5 Points)

Consider the double shuffle space

$$\mathrm{DS}_3(d) = \left\{ f \in \mathbb{Q}[x_1,x_2,x_3]_{(d)} \;\middle|\; f|_{\sqcup\!\sqcup(1,2)} = f^{\#}|_{\sqcup\!\sqcup(1,2)} = 0 \right\}$$

and the following sequence

$$0 \to \mathrm{DS}_3(n) \xrightarrow{i} \mathbb{Q}[x_1,x_2,x_3]^H_{(n)} \xrightarrow{\pi} \mathbb{Q}[x_1,x_2,x_3]^G_{(n)} \oplus \mathbb{Q}[x_1,x_2,x_3]^{G^p}_{(n)} \to 0\,, \qquad (*)$$

where $H = \langle t, ptp^{-1}, -\mathrm{id}\rangle$, with $t = \left(\begin{smallmatrix} 0&0&1\\0&1&0\\1&0&0 \end{smallmatrix}\right)$, $p^{-1} = \left(\begin{smallmatrix} 1&0&0\\1&1&0\\1&1&1 \end{smallmatrix}\right)$. Moreover, $G = \langle t, ptp^{-1}, c_3\rangle$ and $G^p = \langle ptp^{-1}, t, pc_3p^{-1}\rangle$, where $c_3 = \left(\begin{smallmatrix} 0&1&0\\0&0&1\\1&0&0 \end{smallmatrix}\right)$, and $\pi(f) = f|_{\sqcup\!\sqcup(1,2)} \oplus f|_{p\sqcup\!\sqcup(1,2)p^{-1}}$.

a) Using Molien's theorem (Ex 2.1), compute the Molien series of $\mathbb{Q}[x_1,x_2,x_3]^H_{(n)}$, of $\mathbb{Q}[x_1,x_2,x_3]^G_{(n)}$ and of $\mathbb{Q}[x_1,x_2,x_3]^{G^p}_{(n)}$. Assuming $(*)$ is exact, show that

$$\sum_{n\geq 0} \dim_{\mathbb{Q}} \mathrm{DS}_3(n) t^n = 1 + \frac{t^8(1+t^2-t^4)}{(1-t^2)(1-t^4)(1-t^6)} = 1 + \sum_{\substack{n>0 \\ \text{even}}} \left\lfloor \frac{n^2-1}{48} \right\rfloor t^n\,.$$

Using that $g_{k,\ell} \leq \dim_{\mathbb{Q}} \mathrm{DS}_\ell(k-\ell)$ from Ihara-Kaneko-Zagier, give a bound on $g_{k,3}$.

b) Show that the map π is well-defined, and that $\ker(\pi) = \mathrm{DS}_3(n)$.

Hint: Write the index shuffle operation $\sqcup\!\sqcup(1,2) = 1 + c_2 + c_3$, where $c_2 = \left(\begin{smallmatrix} 0&1&0\\1&0&0\\0&0&1 \end{smallmatrix}\right)$.

c) By considering the dual spaces, show that π is surjective if and only if $\mathbb{Q}[x_1,x_2,x_3]^G_{(n)} \cap \mathbb{Q}[x_1,x_2,x_3]^{G^p}_{(n)} = 0$. Using the hint below, conclude that π is surjective.

Hint: Suppose that $\Gamma \subset \mathrm{GL}_n(\mathbb{Z})$ has finite index, and $f\colon \mathbb{R}^n \to \mathbb{R}$ is a Γ-invariant function. Then the function f is constant. (Can you indicate why?)

Keywords for the week 15.06.20–21.06.20: Hilbert series and Hilbert polynomial, Hilbert-Poincaré series, Möbius inversion

Exercise 8.1: (5 points)

a*) Let $\mu(n)$ denote the Möbius function defined by

$$\mu(n) = \begin{cases} 0 & \text{if a square divides } n, \\ -1 & \text{if } n = \pm p_1 \cdots p_r \text{ with } r \text{ odd}, \\ +1 & \text{if } n = \pm p_1 \cdots p_r \text{ with } r \text{ even}. \end{cases}$$

Show that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

b) Let $(f_i)_{i=1}^\infty, (g_i)_{i=1}^\infty$ be two sequences, which satisfy $g_n = \sum_{d|n} f_d$. Using the result in part a), show the Möbius inversion formula holds $f_n = \sum_{d|n} \mu(d) g_{n/d}$.

c*) Let $\Phi_d(x)$ be the d-th cyclotomic polynomial, i.e. $\Phi_d(x)$ is the minimal polynomial of the primitive d-th root of unity $\zeta_d = \exp(2\pi i/d)$. (All other primitive d-th roots of unity are roots of $\Phi_d(x)$. Why? How many primitive d-th roots of unity are there?)

Let n be a positive integer. Show that $\prod_{d|n} \Phi_d(x) = x^n - 1$. Use Möbius inversion to give an explicit formula for Φ_n (involving polynomial multiplication and division).

Hint: Take logarithms.

Exercise 8.2: (5 points)

a) Let A, B be graded K-algebras. Show the following identities of Hilbert-Poincaré series

$$H_{A\oplus B}(t) = H_A(t) + H_B(t)\,,$$
$$H_{A\otimes B}(t) = H_A(t) \cdot H_B(t)\,,$$

where $A \oplus B$ and $A \otimes B$ are define through $(A \oplus B)_k = A_k \oplus B_k$ and $(A \otimes B)_k = \sum_{i+j=k} A_i \otimes B_j$, respectively.

b) Compute $H_A(t)$ for the polynomial algebras

$$A = K[x]\,, \quad A = K[x_1, \ldots, x_n]\,, \quad A = K[f_1, \ldots, f_n]$$

where grading is given by degree of the polynomial, and $f_1, \ldots, f_n \in K[x_1, \ldots, x_n] \setminus K$ are non-constant homogeneous polynomials.

c) Let $\mathcal{A} = k\langle f_1, \ldots, f_n \rangle$ be a free non-commutative polynomial algebra generated by elements f_i in degree $\deg(f_i)$. Show that

$$H_{\mathcal{A}}(t) = \frac{1}{1 - \sum_{i=1}^n t^{\deg(f_i)}}\,.$$

d) Let $\mathcal{A} = \mathrm{Sym}(V)$ be the symmetric algebra on a (graded) vector space V, i.e. A is the free polynomial algebra on a basis of V. Show that the Hilbert-Poincaré series of $\mathcal{A}$ satisfies

$$H_{\mathcal{A}}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{H_V(t^n)}{n}\right).$$

e*) Using the Hilbert-Serre Theorem, indicate how to prove

$$H_{K[x_1,x_2]/I} = \frac{1 - t^{\deg(f)} - t^{\deg(g)} + t^{\deg(f)+\deg(g)-\deg(\gcd(f,g))}}{(1-t)^2},$$

where $I = (f,g)$, with $f, g \in K[x_1,x_2] \setminus K$ non-constant homogeneous polynomials. Investigate the differences in dimensions for $f = xy, g = x^2 \in K[x,y]$ and $f = xy, g = x^2 + y^2 \in K[x,y]$.

Hint: Hint forthcoming

Exercise 8.3: (5 points)

a) Suppose A is a connected graded free $\mathbb{Q}$-algebra, with g_k algebra generators in degree k. Show that

$$H_A(t) = \prod_{k\geq 1}(1-t^k)^{-g_k}.$$

b) Let c_k be the coefficients of $\log H_A(t)$. By taking logarithms of the result in part a) and using Möbius inversion, show that

$$g_k = \sum_{d|k} \frac{\mu(d)}{d} c_{k/d}$$

c) According to Zagier's conjecture, the algebra of $\mathcal{Z}$ of MZV's has Hilbert-Poincaré series

$$H_{\mathcal{Z}}(t) = \frac{1}{1-t^2-t^3}$$

According to the standard conjectures on MZV's $\mathcal{Z}$ is connected graded free algebra. Assuming this, use part b) to show the (conjectural) number of algebra generators $g_k^{\mathcal{Z}}$ of $\mathcal{Z}$ is given by

$$g_k^{\mathcal{Z}} = \frac{1}{k}\sum_{d|k} \mu(k/d) p_d$$

where $p_d = p_{d-2} + p_{d-3}$, $d \geq 4$, with $p_1 = 0, p_2 = 2, p_3 = 3$. Compute $g_k^{\mathcal{Z}}$ for $1 \leq k \leq 20$, with computer assistance.

d*) Check that

$$H_{\mathcal{Z}}(t) = \frac{1}{1-t^2} \cdot \frac{1}{1-t^3-t^5-t^7-t^9-t^{11}-\ldots},$$

and explain what interpretation this suggests for the structure of $\mathcal{Z}$.

Keywords for the week 08.06.20–14.06.20: Generating series, Iterated integral, Regularisation

Exercise 7.1: (5 points)
(Moved from Week 6.) Let

$$F_\ell(t_1,\ldots,t_\ell) = \sum_{(k_1,\ldots,k_\ell)\in(\mathbb{Z}_{>0})^\ell} x_0^{k_1-1}x_1\cdots x_0^{k_\ell-1}x_1 \cdot t_1^{k_1-1}\cdots t_\ell^{k_\ell-1}$$

be the generating series of depth ℓ words. Compute the following shuffle products of generating series, and express them in terms of $F_{\ell'}$.

i) $F_1(t_1) ⧢ F_1(t_2)$,

ii) $F_1(t_1) ⧢ F_2(t_2,t_3)$, and

iii) $F_2(t_1,t_2) ⧢ F_2(t_3,t_4)$.

Exercise 7.2: (5 points)
Check the details of the proof that the $F_\ell^\#$ generating series satisfy the shuffle product relation

$$F_r^\#(t_1,\ldots,t_r) ⧢ F_s^\#(t_{r+1},\ldots,t_{r+s}) = F_{r+s}^\#(t_1,\ldots,t_{r+s})|_{⧢(r,s)}\,.$$

a) Let $F_\ell(t_1,\ldots,t_\ell)$ be the generating series of depth ℓ words as in Ex 7.1 above, and let

$$F_\ell^\#(t_1,\ldots,t_r) = F_l(t_1+\cdots+t_\ell, t_2+\cdots+t_\ell,\ldots,t_\ell)\,.$$

Compute explicitly the shuffle product of generating series $F_1^\#(t_1)⧢F_2^\#(t_2,t_3)$, and express it in terms of $F_{\ell'}$ and in terms of $F_\ell^\#$.

b*) Show that

$$\begin{aligned}F_r(t_1,\ldots,t_r) ⧢ F_s(t_{r+1},\ldots,t_{r+s}) &= F_1(t_1+t_{r+1})(F_{r-1}(t_2,\ldots,t_r) ⧢ F_s(t_{r+1},\ldots,t_{r+s}))) \\ &\quad + F_1(t_1+t_{r+1})(F_r(t_1,\ldots,t_r) ⧢ F_{s-1}(t_{r+2},\ldots,t_{r+s})))\,.\end{aligned}$$

Hint: Check that

$$\begin{aligned}F_r(t_1,\ldots,t_r) &= x_1F_{r-1}(t_2,\ldots,t_r) + x_0t_1F_r(t_1,\ldots,t_r) \\ &= (x_1 + x_0t_1F_1(t_1))F_{r-1}(t_2,\ldots,t_r)\,,\end{aligned}$$

and use the result $F_1(t_1) ⧢ F_2(t_2) = F_2^\#(t_1,t_2)|_{⧢(1,1)}$.

c*) Verify in the case $r=1, s=2$ that

$$\begin{aligned}&F_1(t_1+\cdots+t_{r+s})(F_{r+s-1}^\#(t_1,\ldots,t_r;t_{r+2},\ldots,t_{r+s})|_{⧢(r,s-1)} \\ &+ F_{r+s-1}^\#(t_2,\ldots,t_r;t_{r+1},t_{r+2},\ldots,t_{r+s})|_{⧢(r-1,s)}) = F_{r+s}(t_1,\ldots,t_{r+s})|_{⧢(r,s)}\,.\end{aligned}$$

Exercise 7.3: (5 points)

a) Prove that

$$\int\limits_{a>x_1>\cdots>x_r>b} \cdots \int \frac{dx_1}{x_1} \cdots \frac{dx_r}{x_r} = \frac{1}{r!} \log\left(\frac{a}{b}\right)^r$$

b) Let

$$\zeta_\varepsilon(s_1,\ldots,s_k) = \int\limits_{1-\varepsilon>t_1>\cdots>t_{s_1+\cdots+s_k}>0} \cdots \int \overbrace{\frac{dt_1}{t_1} \cdots \frac{dt_{s_1-1}}{t_{s_1-1}}}^{s_1-1} \frac{dt_{s_1}}{1-t_{s_1}} \cdots \cdot$$
$$\cdot \underbrace{\frac{dt_{s_1+\cdots+s_{k-1}+1}}{t_{s_1+\cdots+s_{k-1}+1}} \cdots \frac{dt_{s_1+\cdots+s_k-1}}{t_{s_1+\cdots+s_k-1}}}_{s_k-1} \frac{dt_{s_1+\cdots+s_k}}{1-t_{s_1+\cdots+s_k}},$$

as in the Kontsevich integral representation of $\zeta(s_1,\ldots,s_k)$. Use a) to express $\zeta_\varepsilon(1,1,2)$.

c) Let

$$F_{Ш,\varepsilon}(x_1,\ldots,x_n) = \sum_{s_1,\ldots,s_n\geq 1} \zeta_\varepsilon(s_1,\ldots,s_n) x_1^{s_1-1} \cdots x_n^{s_n-1},$$

be the generating series of 'approximate' multiple zeta values and

$$F^{\#}_{Ш,\varepsilon}(x_1,\ldots,x_n) = F_{Ш,\varepsilon}(x_1+x_2+\cdots+x_n, x_2+\cdots+x_n,\ldots,x_n).$$

Check explicitly that

$$F^{\#}_{1,Ш,\varepsilon}(t_1) F^{\#}_{2,Ш,\varepsilon}(t_2,t_3) = F^{\#}_{3,Ш,\varepsilon}(t_1,t_2,t_3)|_{Ш(1,2)}.$$

Exercise 7.4: (5 points)

a) Let $A(u) = e^{\gamma u}\Gamma(1+u)$, where $\gamma = 0.577\ldots$ is the Euler-Mascheroni constant and $\Gamma(t)$ is the Gamma function. Show that

$$A(u)^{-1} = 1 + \sum_{\ell\geq 1} \zeta_*^{T=0}(\{1\}^\ell) u^\ell.$$

b) Expand $A(u) = \sum_{k\geq 0} \gamma_k u^k$, and compute γ_4 and γ_6.

Keywords for the week 25.05–31.05: Radford's Theorem, Lyndon words, Iterated integral, shuffle product, shuffle algebra.

Exercise 6.1: (5 points)

Prove that the map of $(\mathfrak{H}^0, ⧢)$-algebras

$$\begin{aligned} \mathfrak{H}^0[T,U] &\to \mathfrak{H} \\ T &\mapsto x_0 \\ U &\mapsto x_1 \end{aligned}$$

is an isomorphism, so that $\mathfrak{H} = \mathbb{Q}\langle x_0, x_1\rangle$ is a 2-variable polynomial algebra over the admissible words $\mathfrak{H}^0 = \mathbb{Q} + x_0\mathfrak{H}x_1$.

Exercise 6.2: (5 points)

Let $S_{n,j}$, with $p+q=n$ and $\min(p,q) \geq j$, denote the set of words in $(x_0x_1)^p ⧢ (x_0x_1)^q$ containing the subword x_0^2 exactly j times, not counting multiplicity.

For example,

$$\begin{aligned} &(x_0x_1)^3 ⧢ (x_0x_1)^1 = 4(x_0x_1\,x_0x_1\,x_0x_1\,x_0x_1) \\ &+ 4(x_0^2x_1\,x_0x_1\,x_0x_1\,x_1) + 4(x_0^2x_1\,x_0x_1\,x_1\,x_0x_1) + 4(x_0^2x_1\,x_1\,x_0x_1\,x_0x_1) \\ &+ 4(x_0x_1\,x_0^2x_1\,x_0x_1\,x_1) + 4(x_0x_1\,x_0^2x_1\,x_1\,x_0x_1) + 4(x_0x_1\,x_0x_1\,x_0^2x_1\,x_1)\,. \end{aligned}$$

So $S_{4,0}$ is given by the first line, while $S_{4,1}$ is given by the second and third lines. However for $S_{4,2}$ we need to consider $(x_0x_1)^2 ⧢ (x_0x_1)^2$. (Does $S_{n,j}$ depend on the decomposition of $n = p+q$? What is the cardinality of $S_{p+q,j}$?)

a) Show that

$$(x_0x_1)^p ⧢ (x_0x_1)^q = \sum_{j=0}^{\min(p,q)} 4^j \binom{p+q-2j}{p-j} \Bigg(\sum_{w \in S_{p+q,j}} w \Bigg)\,.$$

b) Use the above to show

$$\sum_{r=-n}^{n} (-1)^r \Big[(x_0x_1)^{n-r} ⧢ (x_0x_1)^{n+r} \Big] = 4^n \left(x_0^2x_1^2\right)^n\,.$$

c) Use the above, and the evaluation $\zeta(\{2\}^r) = \frac{\pi^{2r}}{(2r+1)!}$ to give a formula for $\zeta(\{3,1\}^n)$ in terms of π^{4n}.

Exercise 6.3: (5 Punkte)

(Additional)

a) Calculate $\zeta_*^T(1,1,1)$ and $\zeta_{⧢}^T(1,1,1)$.

b) Prove that for admissible $\mathbf{s}$

$$\zeta_*(\{1\}^n\mathbf{s}) = \zeta(\mathbf{s})\frac{T^n}{n!} + \text{lower order terms}\,, \text{ and}$$
$$\zeta_{ш}(\{1\}^n\mathbf{s}) = \zeta(\mathbf{s})\frac{T^n}{n!} + \text{lower order terms}\,.$$

Exercise 6.4***:** (5 points)

Let $\mathcal{A} = (\mathbb{Q}\langle A\rangle, ш)$ be the shuffle algebra over some set of letters with an order $A = \{a_0 < a_1 < \cdots < a_k\}$. We call a word $w \neq 1 \in A^*$ a *Lyndon word* if whenever $w = uv$, with $u, v \in A^* \setminus \{1\}$, we have $w < v$ in the induced lexicographic order on A^*.

a) Compute the Lyndon words of length ≤ 4, for $A = \{x_0 < x_1\}$.

b) Show that the following are equivalent characterisations of Lyndon words.

- w is the *unique* minimal element (in the lexicographic ordering) of all non-trivial rotations of w. (A rotation of $w = x_1 x_2 \cdots x_n$ means a word of the form $x_i x_{i+1} \cdots x_n x_1 \cdots x_{i-1}$, $i = 1, \ldots, n$.)
- If $w = uv$, with $u, v \in A^* \setminus \{1\}$, then $u < v$.

c) Suppose $w = \ell_1^{s_1} \ell_2^{s_2} \cdots \ell_k^{s_k}$ is a factorisation of w into a concatenation of Lyndon words with $\ell_1 > \ell_2 > \cdots > \ell_k$ of maximal length. Show that

$$\ell_1^{ш s_1} ш \cdots ш \, \ell_k^{ш s_k} = (s_1! \cdots s_k!) w + \sum_{u < w} \alpha_u u$$

for some coefficients α_u.

d) Let $L = \{\ell \mid \ell \in A^* \text{ is a Lyndon word}\}$. Use b) to prove that the Lyndon words are algebraically independent, and hence that L is a polynomial basis for $\mathcal{A}$.

Hint: This is Radford's Theorem.

Keywords for the week 18.05-24.05: Alphabet, Free non-commutative algebra, Regularisation, Shuffle, Quasi-shuffle, Hoffman isomorphism.

Additional background: Michael E Hoffman and Kentaro Ihara. *Quasi-shuffle products revisited* (2017)

Exercise 5.1⋆: (5 points)
Let

$$\mathrm{St}(\ell,\ell';r)=\begin{Bmatrix}\sigma\colon \{1,2,\dots,\ell+\ell'\}\twoheadrightarrow\{1,2,\dots,\ell+\ell'-r\}\text{ surjective,}\\ \sigma(1)<\sigma(2)<\cdots<\sigma(\ell)\text{ and }\sigma(\ell+1)<\sigma(\ell+2)<\cdots<\sigma(\ell+\ell')\end{Bmatrix}.$$

Show the following stuffle product expression is well-defined and correct

$$\zeta(s)\zeta(s')=\sum_{r=0}^{\min(\ell(s),\ell(s'))}\sum_{\substack{\sigma\in\\ \mathrm{St}(\ell(s),\ell(s');r)}}\zeta(s''(\sigma)_1,\dots,s''(\sigma)_{\ell+\ell'-r}),$$

where

$$s''(\sigma)_k=\begin{cases}s_i & \text{if } \sigma^{-1}(k)=\{i\},\ i\le\ell\\ s'_j & \text{if } \sigma^{-1}(k)=\{\ell+j\}\\ s_i+s'_j & \text{if } \sigma^{-1}(k)=\{i,\ell+j\},\ i\le\ell\,.\end{cases}$$

Exercise 5.2: (5 points)
Prove that the quasi-shuffle product $*_\diamond\colon \mathbb{Q}\langle A\rangle\times\mathbb{Q}\langle A\rangle\to\mathbb{Q}\langle A\rangle$ is associative.

Exercise 5.3: (5 points)
Compute

a) $y_a y_b * y_c$,

b) $y_a y_b * y_c y_d$ and,

c) $y_a y_b y_c * y_d y_e$.

Exercise 5.4⋆: (5 points)
Using the notation from lectures, let $w=y_1^m w_0\in\mathfrak{H}^1$ with $m\ge 0$ and $w_0\in\mathfrak{H}^0$. Show that

a)
$$\operatorname{reg}_*(w) = \sum_{i=0}^{m} \frac{(-1)^i}{i!} y_1^{*i} * y_1^{m-i} w_0$$

b)
$$w = \sum_{i=0}^{m} \frac{1}{i!} \operatorname{reg}_*(y_1^{m-i} w_0) * y_1^{*i}$$

c)
$$\operatorname{reg}_*^T \left(\frac{1}{1 - y_1 u} w_0 \right) = \operatorname{reg}_* \left(\frac{1}{1 - y_1 u} w_0 \right) e^{Tu} = \left(\exp_*(-y_1 u) * \frac{1}{1 - y_1 u} w_0 \right) e^{Tu}$$

Exercise 5.5: (5 points)
Calculate the stuffle-polynomials and stuffle-regularised multiple zeta values

a) $\zeta_*(1, k)$ and

b) $\zeta_*(1, 1, k)$.

Check that a) matches with the previous calculation seen in the lectures via the depth 2 generating series.

Exercise 5.6: (5 points)
Let $Y = \{y_1, y_2, \ldots, \}$ be a countable alphabet and let $\sqcup\!\sqcup = *_\diamond$, induced from the 'zero-product' $y_i \diamond y_j = 0$ on $\mathbb{Q}Y$. Proceed as follows to show that $(\mathbb{Q}\langle Y \rangle, \sqcup\!\sqcup) \cong (\mathbb{Q}\langle Y \rangle, *_\blacklozenge)$, where $*_\blacklozenge$ is some quasi-stuffle product induced from $y_i \blacklozenge y_j$).

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a composition of ℓ (i.e. $\lambda_1 + \cdots + \lambda_n = \ell$, and the order of $\lambda_1, \ldots, \lambda_n$ is important). We write $\mathcal{C}(\ell)$ for the set of all such compositions. For a word $w = a_1 a_2 \ldots a_\ell \in \mathbb{Q}\langle Y \rangle$ of length $\ell(w) = \ell$, λ acts on w by

$$\lambda[w] = \underbrace{[a_1, a_2, \ldots, a_{i_1}]}_{\text{first } i_1 \text{ terms}} \underbrace{[a_{i_1+1}, a_{i_1+2}, \ldots, a_{i_1+i_2}]}_{\text{next } i_2 \text{ terms}} \cdots \underbrace{[a_{i_1+\cdots+i_{n-1}}, a_{i_1+\cdots+i_{n-1}+1}, \ldots, a_\ell]}_{\text{last } i_n \text{ terms}},$$

where $[a_1, a_2, \ldots, a_n] = a_1 \blacklozenge a_2 \blacklozenge \cdots \blacklozenge a_n$. Define

$$\exp(w) = \sum_{(\lambda_1, \ldots, \lambda_n) \in \mathcal{C}(\ell(w))} \frac{1}{\lambda_1! \cdots \lambda_n!} \lambda[w]$$

and

$$\log(w) = \sum_{(\lambda_1, \ldots, \lambda_n) \in \mathcal{C}(\ell(w))} \frac{(-1)^{\ell(w)-n}}{\lambda_1 \cdots \lambda_n} \lambda[w]$$

and extend this by linearity to $\mathbb{Q}\langle Y \rangle$. (Note both functions use $\blacklozenge$.)

The goal is to show $\exp(w)$ is an isomorphism from $(\mathbb{Q}\langle Y \rangle, \sqcup\!\sqcup) \cong (\mathbb{Q}\langle Y \rangle, *_\blacklozenge)$, with inverse given by $\log(w)$.

a) Compute $\exp(w)$ and $\log(w)$ for $w = y_{i_1} y_{i_2}$, $w = y_{i_1} y_{i_2} y_{i_3}$ and $w = y_{i_1} y_{i_2} y_{i_3} y_{i_4}$. Check that $\log(\exp(w)) = w$ and $\exp(\log(w)) = w$.

b) Compute $\exp(y_{i_1} y_{i_2} \sqcup\!\sqcup y_{i_3})$ and compare with $\exp(y_{i_1} y_{i_2}) *_\blacklozenge \exp(y_{i_3})$, similarly compute and compare $\exp(y_{i_1} y_{i_2} y_{i_3} \sqcup\!\sqcup y_{i_4}) \stackrel{?}{=} \exp(y_{i_1} y_{i_2} y_{i_3}) *_\blacklozenge \exp(y_{i_4})$ and $\exp(y_{i_1} y_{i_2} y_{i_3} \sqcup\!\sqcup y_{i_4} y_{i_5}) \stackrel{?}{=} \exp(y_{i_1} y_{i_2} y_{i_3}) *_\blacklozenge \exp(y_{i_4} y_{i_5})$.

c*) Show generally that $\exp \circ \log = \mathrm{id}$ and $\log \circ \exp = \mathrm{id}$.

Hint: If $f(x) = a_1 x + \sum_{i\geq 2} a_i x^i$, with $\circ$-inverse $f^{-1}(x) = b_1 x + \sum_{i\geq 2} b_i x^i$, show that

$$\Psi_f(w) = \sum_{(\lambda_1,\ldots,\lambda_n)\in\mathcal{C}(\ell(w))} a_{\lambda_1}\cdots a_{\lambda_\ell}\lambda[w] \text{ and } \Psi_{f^{-1}}(w) = \sum_{(\lambda_1,\ldots,\lambda_n)\in\mathcal{C}(\ell(w))} b_{\lambda_1}\cdots b_{\lambda_\ell}\lambda[w]$$

are also inverses. How does the coefficient of $\mu[w]$, $\mu \in \mathcal{C}(\ell(w))$, in $\Psi_f \circ \Psi_{f_{-1}}$ arise?

d*) Show generally that $\exp(w \sqcup\!\sqcup w') = \exp(w) *_{\blacklozenge} \exp(w')$.

Hint: Each side is a sum of rational multiples of $[S_1, T_1][S_2, T_2]\cdots[S_k, T_k]$, where each S_i a subseuqence of w and each T_j is a subsequence of w'. Compare the coefficient of this on each side.

From part c) and d) we conclude that $\exp(w)$ is an isomorphism.

Keywords for the week 11.05-17.05: (Extended) period polynomials.

Additional background: Winfried Kohnen and Don Zagier. *Modular forms with rational periods* (1984)

Recall $V_k = \operatorname{span}_{\mathbb{Q}} \left\{ X_1^{r-1} X_2^{s-1} \mid r+s=k, r,s \geq 1 \right\}$ is the space of two-variable, degree $k-2$ polynomials.

Exercise 4.1: (5 points)

Assume k is even, and define the pairing $\langle \cdot, \cdot \rangle \colon V_k \times V_k \to \mathbb{Q}$, by

$$\langle F(X_1, X_2), G(X_1, X_2) \rangle = -\frac{1}{(k-2)!} F\left(-\tfrac{\partial}{\partial X_2}, \tfrac{\partial}{\partial X_1}\right)\Big(G(X_1, X_2)\Big)\Big|_{(0,0)}$$

Check that this is a non-degenerate bilinear $\mathrm{PGL}_2(\mathbb{Z})$-invariant pairing (i.e. $\langle F|_\gamma, G|_\gamma \rangle = \langle F, G \rangle$, for $\gamma \in \mathrm{PGL}_2(\mathbb{Z})$, see Ex 3.2), and that

$$\langle X_1^{r-1} X_2^{s-1}, X_1^{m-1} X_2^{n-1} \rangle = \frac{(-1)^r}{\binom{k-2}{m-1}} \delta_{(r,s)=(n,m)} \, ,$$

where $\delta_\bullet$ is the Kronecker delta (i.e. $\delta_\bullet = 1$ if $\bullet$ is true, and $\delta_\bullet = 0$ if $\bullet$ is false).

Exercise 4.2: (5 points)

a) Use the pairing from Ex 4.1 to show that

$$V_k = V_k^+ \oplus V_k^-$$

is an orthogonal decomposition, where $V_k^\pm = \{ f \in V_k \mid f|_\varepsilon = \pm f \}$ is the ± 1-eigenspace of ε. (See Ex 3.3.)

b) Show also that

$$V_k = V_k^{\mathrm{ev},+} \oplus V_k^{\mathrm{ev},-} \oplus V_k^{\mathrm{od},+} \oplus V_k^{\mathrm{od},-}$$

is an orthogonal decomposition, where V_k^{ev} and V_k^{od} are the $(+1)$- and (-1)-eigenspaces of δ, respectively and $V_k^{\mathrm{ev},\pm} = V_k^{\mathrm{ev}} \cap V_k^\pm$ and $V_k^{\mathrm{od},\pm} = V_k^{\mathrm{od}} \cap V_k^\pm$

c) Use this to decompose $X_2^2, X_2^4, X_2^6, X_2^{10}$ according to the description of V_k in b). Can you find a general formula for X_2^{k-2}?

Exercise 4.3: (5 points)

Use Molien's Theorem from Ex. 2.1, to compute the dimensions of V_k^+, V_k^{ev} and $V_k^{T^{-1}\delta}$, where $V_k^+ = V_k^\varepsilon$ is the $+1$-eigenspace of ε (see Ex 3.3) and

$$V_k^M = \{ f \in V_k \mid f|_M = f \} \, .$$

b) Calculate the dimensions of $V_k^{T,\varepsilon}$ and $V_k^{T,\delta}$, where

$$V_k^{M_1,M_2} = \{ f \in V_k \mid f|_{M_1} = f|_{M_2} = f \} \, .$$

Exercise 4.4: (5 points)

a) Give formulae for

$$\zeta(\mathrm{ev},\mathrm{ev}) = \sum_{i,j \text{ odd}} a_{i,j}\zeta(i,j)$$

in the cases $\zeta(2,2)$, $\zeta(4,2)$, $\zeta(2,4)$, $\zeta(6,2)$, $\zeta(4,4)$ and $\zeta(2,6)$.

b*) Give a general formula for $\zeta(\mathrm{ev},\mathrm{ev})$ in terms of $\zeta(\mathrm{od},\mathrm{od})$.

Exercise 4.5*: (5 points)

Let V be a vector space over a field K. Suppose that $\langle\cdot,\cdot\rangle\colon V\times V\to K$ is a non-degenerate pairing. Show that $\mathrm{Hom}(V,K)\cong\{\langle v,\cdot\rangle\colon V\to K \mid v\in V\}$.

Exercise 4.6: (5 points)

a) For $k=4,6,8,12$, check that $p_k := 2\big((X_2^{k-2})|_{1+\varepsilon-ST}\big)|_{T^{-1}-1}$ gives the relation

$$4\sum_{i=1}^{(k-2)/2} a(2i+1,k-2i-1) = a(k) \tag{$*$}$$

as claimed in lectures by computing $\langle(X_2^{k-2})|_{1+\varepsilon-ST}),\mathcal{R}_A\rangle$, and

$$\langle p_k,\mathcal{A}\rangle = \langle(X_2^{k-2})|_{1+\varepsilon-ST}\big)|_{\Delta^\star},\mathcal{A}\rangle = \langle(X_2^{k-2})|_{1+\varepsilon-ST}\big),\mathcal{A}|_\Delta\rangle$$

where $\Delta^\star=(1+\varepsilon)(T^{-1}-1)$ is the adjoint to Δ.

b*) Give a general proof that p_k gives the relation $(*)$

Exercise 4.7*: (5 points)

Write $V_k^{T^{-1}\delta,\mathrm{sym}} = \{f\in V_k^{T^{-1}\delta} \mid f|_{(1-\delta)(1-\varepsilon)}=0\}$.

a) Check that $0\to V_k^{T^{-1}\delta,\varepsilon}\to V^{T^{-1}\delta,\mathrm{sym}}\xrightarrow{1-\varepsilon} W_k^-\to 0$ is a short exact sequence for $k=4,6,8$ and 12.

Hint: Make use of a computer algebra systems.

b) Show that we have a splitting

$$0\longrightarrow V_k^{T^{-1}\delta,\varepsilon}\longrightarrow V^{T^{-1}\delta,\mathrm{sym}} \underset{\frac{1}{2}(T+\delta)}{\overset{1-\varepsilon}{\rightleftarrows}} W_k^-\longrightarrow 0$$

Keywords for the week 04.05–10.05: Bernoulli numbers, Slash operator, (Extended) period polynomials, Period, Power series Ansatz

Exercise 3.1: (5 points)

Define the sequence $\{q_{2n}\}_n$ recursively by $q_2 = 1$, and

$$q_{2n} = \frac{2}{2n+1} \sum_{k=1}^{n-1} q_{2k} q_{2(n-k)} \,, \qquad n > 1 \,.$$

Prove that

$$q_{2n} = (-1)^{n-1} \frac{24^n}{2(2n)!} B_{2n} \,.$$

where B_k is the k-th Bernoulli number.

Exercise 3.2: (5 points)

Given a function $f(X_1, X_2)$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we set $f|_\gamma(X_1, X_2) = f(aX_1 + bX_2, cX_1 + dX_2)$. Show that if $f \in V_k = \mathrm{span}_{\mathbb{Q}} \left\{ X_1^{r-1} X_2^{s-1} \mid r + s = k, r, s \geq 1 \right\}$, degree $k-2$ homogeneous polynomials, and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $f|_\gamma$ defines a group action. If k is even, $f \in V_k$, and $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$, show that $f|_\gamma$ also defines a group action.

Exercise 3.3⋆: (5 points)

Set

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
$$\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T' := U^2 S \,.$$

Show the following identities hold (modulo ± 1):

$$\begin{aligned} S^2 \equiv U^3 \equiv 1 \,, \qquad & T \equiv US \,, \qquad & S \equiv \varepsilon\delta \,, \\ \delta = S\varepsilon \equiv \varepsilon S \,, \qquad & U^2 \equiv \varepsilon U \varepsilon \,, \qquad & T^{-1} = \delta T \delta \,, \\ T' \equiv \varepsilon T \varepsilon \,, \qquad & (T')^{-1} = STS \,, \qquad & T^{-1}\varepsilon T = ST\varepsilon \end{aligned}$$

Exercise 3.4: (5 points)

Set

$$\begin{aligned} W_k &:= \{ f \in V_k \mid f|_{1+S} = f|_{1+U+U^2} = 0 \} \,, \text{ and} \\ W_k^\pm &= W_k \cap V_k^\pm \,, \end{aligned}$$

where $V_k^\pm = \{ f \in V_k \mid f|_\varepsilon = \pm f \}$ is the ± 1-eigenspace of ε.

a) Show that $W_k = \{ f \in V_k \mid f|_{1-T-T'} = 0 \}$.

b) Show that $W_k^\pm = \{ f \in V_k \mid f|_{1-T \mp T\varepsilon} = 0 \}$.

c*) Indicate why $W_k^+ \cong S_k$ and $W_k^- \cong M_k$, where S_k and M_k are the spaces of cusp forms and modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$ respectively.

d*) Introduce $\widehat{W}_k^+ = \{f \in \widehat{V}_k \mid f|_{1-\varepsilon} = f|_{1+S} = f|_{1+U+U^2} = 0\}$, the space of *extended period polynomials*, where

$$\widehat{V}_k = \mathrm{span}_{\mathbb{Q}} \left\{ X_1^{r-1} X_2^{s-1} \mid r+s=k, r,s \geq 0 \right\} = V_k \oplus \frac{X_1^{k-1}}{X_2}\mathbb{Q} \oplus \frac{X_2^{k-1}}{X_1}\mathbb{Q} .$$

Calculate $\widehat{W}_k^+$ for $2 \leq k \leq 8$, and indicate why $\widehat{W}_k^+ \cong M_k$.

Exercise 3.5*: (5 points)

Consider the differential equation

$$\begin{cases} f'(x) = f(x) , \\ f(0) = 1 . \end{cases}$$

Make the power series Ansatz

$$f(x) = \sum_{i=0}^{\infty} a_i x^i ,$$

and use this to compute the first few coefficients a_i, $0 \leq i \leq 4$, and to find a recursive formula for the coefficients a_i.

Keywords for the week 27.04–03.05: Generating series, Regularisation, Partial fraction expansion, Euler sum formula

Exercise 2.1: (5 points)

a*) Prove Molien's Theorem: Let $G \subset \mathrm{GL}_n(K)$ be a finite group acting on the vector space of polynomials $K[x_1, \ldots, x_n]$ over a field $K \subset \mathbb{C}$ through $(x_1, \ldots, x_n)^\top \mapsto g\,(x_1, \ldots, x_n)^\top$ for $g \in G$. Let $\chi\colon G \to K^\times$ be a character and write

$$K[x_1, \ldots, x_n]_{(k)}^{(G,\chi)} = \left\{ f \in K[x_1, \ldots, x_n] \;\middle|\; \begin{array}{l} f \text{ homogeneous of degree } k, \text{ and} \\ f(g \cdot x) = \chi(g) f(x) \text{ for all } g \in G \end{array} \right\}$$

for the space of (relative) invariant polynomials of degree k. Then we have

$$\sum_{k=0}^{\infty} \dim_K \left(K[x_1, \ldots, x_n]_{(k)}^{(G,\chi)} \right) t^k = \frac{1}{|G|} \sum_{g \in G} \frac{\overline{\chi}(g)}{\det(1 - gt)} \,.$$

b) Apply Molien's Theorem to the double-shuffle spaces

$$DS_2(k) = \left\{ \begin{array}{l} f \in \mathbb{Q}[x,y] \text{ homogeneous} \\ \text{of degree } k \end{array} \;\middle|\; \begin{array}{l} f(x,y) + f(y,x) = 0\,, \\ f(x+y,y) + f(x+y,x) = 0 \end{array} \right\}$$

to compute $\dim_{\mathbb{Q}} DS_2(k)$. Hence obtain the bound $D_{k,\ell} \leq \lfloor \frac{k-2}{6} \rfloor$ from Ihara-Kaneko-Zagier.

Hint: Show that $G = \langle t, p \rangle$ where $t = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $p = \left(\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\right)$ acts on $\mathbb{Q}[x,y]$, and $(\mathbb{Q}[x,y])_{(k)}^{(G,\chi)} = DS_2(k)$. What is χ?.

Exercise 2.2: (5 points)

a) Let K be a field, and $\mathcal{A}_\bullet$ be a connected graded K-algebra, such that each $\mathcal{A}_k$ is finite dimensional; recall connected means $\mathcal{A}_{<0} = 0$ and $\mathcal{A}_0 \cong K$. Set $\mathcal{L}_k = \mathcal{A}_k / (\mathcal{A}_k \cap \mathcal{A}_{>0}^2)$, show that

$$\dim_K \mathcal{L}_k = \text{number of generators of } \mathcal{A} \text{ in degree } k\,.$$

b) Suppose $\mathcal{A}_{\bullet,\bullet}$ is a connected bigraded K-algebra, such that each $\mathcal{A}_{k,\ell}$ is finite dimensional. Set $\mathcal{L}_{k,\ell} = \mathcal{A}_{k,\ell} / (\mathcal{A}_{k,\ell} \cap \mathcal{A}_{\neq(0,0)}^2)$, show that

$$\dim_K \mathcal{L}_{k,\ell} = \text{number of generators of } \mathcal{A} \text{ in degree } (k,\ell)\,.$$

c) Recall that a filtered (connected) graded K-algebra $\mathcal{A}_\bullet$ is one where the filtration is compatible with the grading in the sense that $F_\ell \mathcal{A} = \bigoplus_k F_\ell \mathcal{A}_k$. Set

$$\mathcal{A}_{k,\ell} = \mathrm{gr}_l^F \mathcal{A}_k = F_\ell \mathcal{A}_k / F_{\ell-1} \mathcal{A}_k$$

Show that $\mathcal{A}_{\bullet,\bullet} = \bigoplus_{k,\ell} \mathcal{A}_{k,\ell}$ is a (connected) bigraded algebra.

Exercise 2.3: (5 points)

a) Prove Euler's sum formula using the identities in Ex 2.4, for $p \geq 1$

$$\sum_{n=0}^{p-2} \zeta(p-n, n+1) = \zeta(p+1) \,.$$

b) Reformulate Euler's sum formula to evaluate $\zeta(p, 1)$ in terms of Riemann zeta values. Hence evaluate $\zeta(3,1), \zeta(4,1), \zeta(5,1)$ as polynomials in Riemann zeta values.

c*) Use Nielsen's reduction formula to evaluate all double-zeta values of weight ≤ 7 as polynomials in Riemann zeta values. How far can you get in weight 8? (Don't expect to evaluate everything in weight 8.)

d*) Compare your observations from c*) with the numbers you obtained in Ex 2.1 b).

Exercise 2.4*: (5 points)

a) Fix integers $i, j \geq 1$, prove the partial fractions expansion

$$\frac{1}{X^i Y^j} = \sum_{\substack{r+s=i+j \\ r,s>0}} \binom{r-1}{j-1} \frac{1}{(X+Y)^r X^s} + \binom{r-1}{i-1} \frac{1}{(X+Y)^r Y^s} \,.$$

Hence show the relation

$$\zeta(j)\zeta(k-j) = \sum_{r=2}^{k-1} \left[\binom{r-1}{j-1} + \binom{r-1}{k-j-1} \right] \zeta(r, k-r) \,.$$

b) Check the following partial fractions expansion, for $x \neq a$

$$\frac{1}{x^p (x-a)^q} = (-1)^q \sum_{n=0}^{p-1} \binom{q+n-1}{q-1} \frac{1}{x^{p-n} a^{q+n}} + \sum_{n=0}^{q-1} \binom{p+n-1}{p-1} \frac{(-1)^n}{a^{p+n} (x-a)^{q-n}} \,.$$

Hence show Nielsen's reduction formula, that for $p > 1, q \geq 1$

$$\begin{aligned}\zeta(p,q) = {} & \sum_{n=0}^{q-2} (-1)^n \binom{p+n-1}{p-1} \zeta(q-n)\zeta(p+n) + (-1)^q \sum_{n=0}^{p-2} \binom{q+n-1}{q-1} \zeta(p-n, q+n) \\ & + (-1)^{q-1} \binom{p+q-2}{p-1} \left(\zeta(p+q) + \zeta(p+q-1, 1) \right) .\end{aligned}$$

c) How do the expansions in part a) and b) differ?

Keywords for the week 20.04–26.04:
Riemann zeta function, Bernoulli numbers, Multiple Zeta Values, Stuffle product, Basel problem.

Exercise 1.1*: (5 points)
Sketch a proof of Apéry's theorem.

Hint: The following pages Apéry's constant, Apéry's Theorem, or Section 5.4 of The 1-2-3 of Modular Forms are useful.

Exercise 1.2*: (5 points)
a) Prove that the Taylor expansion of $\log(\Gamma(1-z))$ at $z=0$ is given by

$$\log(\Gamma(1-z)) = \gamma z + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} z^k$$

where γ is the Euler-Mascheroni constant defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n) \right).$$

b) Give examples of other functions whose Taylor coefficients contain zeta values.

Exercise 1.3*: (5 points)
Prove Euler's partial fraction expansion formula for the cotangent:

$$\pi \cot(\pi x) = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n} \right) \qquad (x \in \mathbb{R} \backslash \mathbb{Z}).$$

Exercise 1.4: (5 points)
Show that the numbers d_k in Zagier's Conjecture satisfy

$$\sum_{k\geq 0} d_k x^k = \frac{1}{1-x^2-x^3} \text{ and } \lim_{k\to\infty} \left(d_k - \alpha r^k \right) = 0$$

for some constant α, where $r \approx 1,324717\ldots$ is the real root x^3-x-1.

Exercise 1.5: (5 points)
Prove that if $\mathcal{Z}$ is a graded algebra, then $\zeta(k)$ is transcendental for all $k>1$.

Exercise 1.6: (5 points)
Show that Hoffman's Conjecture implies Zagier's Conjecture if $\mathcal{Z}$ is a graded algebra.

Exercise 1.7: (5 points)

a) Describe the products $\zeta(2)\zeta(2)$, $\zeta(2)\zeta(2,2)$ and $\zeta(5,2)\zeta(3,4)$ as linear combinations of multiple zeta values.

b) Show that $\zeta(p)^2 = 2\zeta(p,p) + \zeta(2p)$, $p > 1$ integer.