Interpretation of relations $<, >, =, \parallel$ on games

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1. Definition of the relations

Recall from the surreal numbers section of the course, $\leq$ is defined recursively and forms the most 'basic' relation we have.

Definition 1.1 ($\leq$)**.** *Suppose we have two games (or surreal numbers) $G = \{ g^L \mid g^R \}$ and $H = \{ h^L \mid h^R \}$. We say $G \leq H$ if and only if no $g^L \geq H$ and $G \geq$ no h^R.*

We then define further relations in terms of this

Definition 1.2.

- *$G \geq H$ iff $H \leq G$,*
- *$G < H$ iff $G \leq H$ and $G \not\geq H$,*
- *$G > H$ iff $G \not\leq H$ and $G \geq H$,*
- *$G = H$ iff $G \leq H$ and $G \geq H$.*

If G is not related to H in any of the above ways, we have $G \parallel H$, and so

- *$G \parallel H$ iff $G \not\leq H$ and $G \not\geq H$.*

2. Interpretation of the relations

Theorem 2.1 (Interpretation)**.** *If $G = \{ g^L \mid g^R \}$ is a game, comparing G to $0 = \{ \mid \}$ tells us the following information about who wins.*

- *$G < 0$ iff right wins,*
- *$G > 0$ iff left wins,*
- *$G = 0$ iff player 2 wins,*
- *$G \parallel 0$ iff player 1 wins.*

Proof. We justify this by induction, dealing with each case in turn.

Case $G < 0$: If $G < 0$, then $G \leq 0$ and $G \not\geq 0$. The former means no $g^L \geq 0$ and $G \geq$ no 0^R. The latter means $0 \not\leq G$, which means some $0^L \geq G$ (no, since $0^L = \emptyset$) or $0 \geq$ some g^R. We want to show that right wins G, by making use of the winning strategies which exist in the parents g^L or g^R.

If right plays first in G, then right can move to some $g^R \leq 0$. If $g^R \geq 0$ holds, then $g^R = 0$ and the new position is a win for the second player. Since it is left's turn to play now, the second player is right. If $g^R \not\geq 0$, then $g^R < 0$ and right wins the new position, regardless of who plays first.

If left plays first in G, then he can only move to some position $g^L \not\geq 0$. If we have $g^L \leq 0$, then $g^L < 0$ and right wins. If instead $g^L \not\leq 0$, then $g^L \parallel 0$, and the first player wins. But in this new position, it is rights turn to play.

The remaining cases are very similar! Check them as an exercise. □