Numbers!

Wintersemester 2016/17

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CHAPTER 1

Introduction and Motivation

Lecture 1 18/10/2016

What, exactly, is a number? How can actually *prove* results about numbers? This question has no easy answer, and touches on many historical and philosophical aspects of mathematics.

The 'natural numbers' $\mathbb{N}$ are maybe the only numbers which did not need to be discovered, having been known since ancient times. Every other system of numbers builds on top of the natural numbers (directly, or indirectly) in order to generalise some desirable/interesting property, or fix some gap/incomplete aspect. The 'classical numbers' $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ all arise by trying to plug gaps with the current numbers, starting with the lack of negatives in $\mathbb{N}$. This process carries on until we arrive at the 'algebraically complete' field $\mathbb{C}$.

In order to say anything coherent during this process, we really ought to know how to rigorously define $\mathbb{N}$. Natural numbers, integers, *et cetera*, satisfy many familiar properties. Addition is commutative, associative, distributes over multiplication. But without knowing how to define $\mathbb{N}$, how can any of these properties be proven?

Alternatively, there are many other ways to extend the natural numbers. Moving from $\mathbb{Q}$ to $\mathbb{R}$ is an example of metric completion, with respect to the Euclidean metric on $\mathbb{Q}$. But we can define difference distances on $\mathbb{Q}$, and complete them to get the p-adic numbers $\mathbb{Q}_p$. The natural numbers can be identified as orderings of points; by extending to infinitely many points we get the ordinal numbers. Or we can use $\mathbb{N}$ to measure sizes, and for infinite sets get the cardinal numbers. We can add infinite or infinitesimal numbers to $\mathbb{R}$ to get the hyperreals. Or we can try to repeat the construction of the complex numbers $\mathbb{C}$, which leads to the Hamiltonians $\mathbb{H}$, octonions $\mathbb{O}$, *et cetera*. Adding infinity in a way which allows an incredible amount of arithmetic to happen leads us to the surreal numbers.

We will spend most of the time studying the surreal numbers **No**. The chapters on surreal numbers are based very much on Conway's book On Numbers and Games, with other inspiration drawn from Knuth's book.

Overview:

- Reminder about the constructions of the classical numbers
- Introduction to surreal numbers, definitions and axioms
- The first few surreal numbers, explicit proofs and properties. Birthday of a surreal number.
- Numbers born on day ω. Irrational numbers. Infinity ω, infinitesimals $\epsilon = 1/\omega$, $\omega - 1$, $\omega/2$, $\sqrt{\omega}$.
- Inductive proofs for surreal numbers. Idea of a day-sum for descent proofs.
- Arithmetic on surreal numbers, addition, multiplication. The surreal numbers form a Field. Properties of addition, multiplication, ordering.

- Relation of surreal numbers, real numbers, ordinal numbers. The simplicity theorem.
- Sign expansion, ω^x map, Conway normal form for a surreal number, irreducible numbers. Gaps in the surreal number line.
- Infinite sums of surreal numbers, analysis with surreals. Surreals are real-closed. Surcomplex numbers $\mathbf{No}[i]$.
- Power series, analytic functions of surreal numbers. Exponential, tangent, logarithm, etc.
- Number theory in the surreals. Omnific integers, continued fractions, Waring's problem.
- Surreal numbers and game theory.
- Open problems, and questions. Calculus on surreal numbers. Problems with genetic functions, problems with integration

CHAPTER 2

The 'classical' numbers: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$

Here we will review the construction of $\mathbb{N}$ using Peano arithmetic, and then see how to extend to the remaining numbers using (the scarily named) Grothendeick group to make $\mathbb{Z}$ out of $\mathbb{N}$, the field of fractions to make $\mathbb{Q}$ out of $\mathbb{Z}$, metric completion to get $\mathbb{R}$ from $\mathbb{Q}$, and algebraic completion to get $\mathbb{C}$ from $\mathbb{R}$.

For the purposes of the course, the two most important constructions are going to be the construction of $\mathbb{N}$ itself, and the use of Dedekind cuts to construct $\mathbb{R}$ from $\mathbb{Q}$. Both of these ideas are, in a sense, generalised when we come to construct the surreal numbers.

1. Peano arithmetic to define $\mathbb{N}$

1.1. The Peano axioms for $\mathbb{N}$. The Peano axioms are used to formally define the object $\mathbb{N}$. By using them, and a formal definition of addition, and multiplication, we will be able to rigorously prove all of the 'familiar' and 'intuitive' properties of natural numbers.

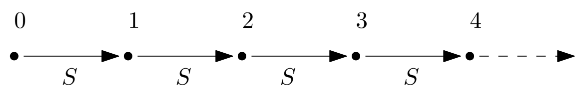
Axiom 2.1 (Peano axioms)**.** There exists a set $\mathbb{N}$, called the set of *natural numbers*, and a function $s\colon \mathbb{N} \to \mathbb{N}$, called the *successor function*. They satisfy the following axioms

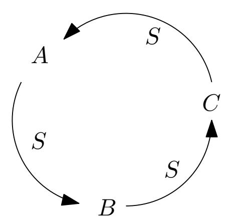
i) 0 is a natural number, that is $0 \in \mathbb{N}$.
ii) For every natural number n, the successor $S(n)$ is a natural number.
iii) For all natural numbers n and m, we have that $S(n) = S(m)$ implies $n = m$. That is, S is injective.
iv) For every natural number n, $S(n) \neq 0$. That is, no natural number has successor 0.

One should think of S, as the 'plus 1' function.

From this we have the number 0. Then the successor of 0 is $S(0)$, which we will call 1. And the successor of 1 is $S(1) = S(S(0))$, which we will call 2.

Remark 2.2. These axioms don't (yet) pin down just the (familiar) natural numbers. For example, the following set satisfies all of these axioms





Without induction, this fits the Peano axioms

To fix this problem, we need some way to eliminate loops, or chains. We need some way to state that S generates all natural numbers.

Axiom 2.3 (Induction for Peano Axioms)**.** The induction axiom allows us to prove a statement is true for all of $\mathbb{N}$, by showing it holds for each successor in turn.

v) Suppose $M \subset \mathbb{N}$ is a set such that $0 \in M$, and for each $m \in M$ we have $S(m) \in M$. Then $M = \mathbb{N}$.

Alternatively

v′) Suppose ϕ is an unary predicate, that is $\phi(n)$ a true/false statement about a natural number n. If $\phi(0)$ is true, and $\phi(n)$ is true implies $\phi(S(n))$ is true. Then ϕ is true for all $n \in \mathbb{N}$.

Exercise 2.4. Show that the two versions of the induction axiom v) and v′) above are equivalent.

Exercise 2.5. Give an example of a formula ϕ which is true for all the 'standard' natural numbers, but fails at A above.

Remark 2.6. For technical reasons, logicians often want to replace the 'second-order' induction axiom with a 'first-order' axiom scheme. Rather than giving an axiom which holds for all predicates, one gives an 'axiom scheme' where each such predicate is written individually.

Namely for each formula $\phi(x, y_1, \ldots, y_k)$, we add the axiom

v″) If, for all $y_1, \ldots, y_k \in \mathbb{N}$, the formula $\phi(0, y, \ldots, y_n)$ holds, and moreover for all n, $\phi(n, y_1, \ldots, y_n)$ implies $\phi(S(n), y_1, \ldots, y_n)$. Then $\phi(n, y_1, \ldots, y_k)$ is true for all natural numbers.

It might seem, at first glance, that the 'first-order' axiom scheme is equivalent to the 'second-order' induction axiom. But, for other technical reasons, it is strictly weaker. Essentially because we can only use finite many axioms at a time, the axiom scheme is not able to uniquely identify the natural numbers. This leads to notions of *non-standard natural numbers.*

1.2. Arithmetic on $\mathbb{N}$.

Definition 2.7 (Addition $+$)**.** Addition of two natural numbers is defined recursively as follows.

$$\begin{aligned} a + 0 &= a \\ a + S(b) &= S(a + b)\,. \end{aligned}$$

Example 2.8. If we call $S(S(S(0))) = 3$, and $S(S(S(S(0)))) = 4$, then we compute

$$\begin{aligned} 3 + 4 &= 3 + S(3) \\ &= S(3 + 3) \\ &= S(3 + S(S(S(0)))) \\ &= S(S(3 + S(S(0)))) \\ &= S(S(S(3 + S(0)))) \\ &= S(S(S(S(3 + 0)))) \end{aligned}$$

$$= S(S(S(S(3))))$$
$$= S(S(S(S(S(S(S(0))))))) .$$

And this last number, we might as well call 7.

We have shown that $3 + 4 = 7$, with these definitions.

Theorem 2.9. *Addition is associative.*

Proof. We want to show that $(a + b) + c = a + (b + c)$, for all natural numbers $a, b, c \in \mathbb{N}$. We will do this by induction using the recursive definition of addition given above.

The induction will happen on c, so we must first establish the result for the base case $c = 0$. This is straight forward. We have

$$(a + b) + 0 = a + b ,$$

using the $+0$ case from the definition of $+$. For exactly the same reason, we have

$$a + (b + 0) = a + (b) = a + b .$$

Combining these two equalities gives

$$(a + b) + 0 = a + (b + 0) ,$$

so the base case $c = 0$ holds.

Now for the inductive step; we assume that the result holds for c, namely

$$(a + b) + c = a + (b + c) ,$$

and we attempt to show the result for $S(c)$. But by the successor case in definition of $+$, we have

$$(a + b) + S(c) = S((a + b) + c) .$$

By the induction hypothesis, we can write this as

$$= S(a + (b + c)) .$$

Now apply the successor case of $+$, to get

$$= a + S(b + c) .$$

Apply the successor case again, to get

$$= a + (b + S(c)) .$$

This shows the statement holds for $S(c)$, so by induction we conclude that addition is associative. □

Theorem 2.10. *Addition is commutative.*

Proof. We want to show that $a + b = b + a$, for all natural numbers $a, b \in \mathbb{N}$. We want to use induction; we have the recursive definition $a + S(b) = S(a + b)$.

We first check the case where $b = 0$. We want to show that $a + 0 = 0 + a$, but this is the case because both sides evaluate to a.

Lemma 2.11. *The natural number 0 is the identity element for $+$. That is $0 + a = a + 0 = a$, for any $a \in \mathbb{N}$.*

Proof. We prove this by induction also. The base case is

$$\underbrace{0}_{a} + 0 = 0 + \underbrace{0}_{a} = 0\,,$$

but this holds by property 1 of the definition of addition. Namely 0 is the right identity of addition.

Now suppose that the statement holds for a. We now show it for $S(a)$. We have (by definition of $+$), that

$$S(a) + 0 = S(a)\,.$$

On the other hand

$$0 + S(a) = S(0 + a) = S(a + 0) = S(a)$$

So the result holds by induction. □

Also we prove explicitly the base case $b = 1$.

Lemma 2.12. *For all natural numbers a, we have $a + 1 = 1 + a$.*

Proof. This is proven by induction, also. The case where $a = 0$ holds because $0 + 1 = 0 + S(0) = S(0 + 0) = S(0) = 1 = 1 + 0$.

Now suppose the result holds for a. Then we get

$$\begin{aligned}
S(a) + 1 &= S(a) + S(0) \\
&= S(S(a) + 0) \\
&= S(S(a + 0) + 0) \\
&= S((a + S(0)) + 0) \\
&= S((a + 1) + 0) \\
&= S(a + 1) \\
&= S(1 + a) \\
&= 1 + S(a)\,.
\end{aligned}$$

Thus the result holds by induction. □

Now we show the result by induction on b. Suppose that the result holds for b, then we show it for $S(b)$.

$$\begin{aligned}
a + S(b) &= a + (b + 1) \\
&= (a + b) + 1 \\
&= (b + a) + 1 \\
&= b + (a + 1) \\
&+ b + (1 + a) \\
&= (b + 1) + a \\
&= S(b) + a\,.
\end{aligned}$$

Thus addition of numbers is also commutative. □

Exercise 2.13. Prove that $\mathbb{N}$ is cancellative, that is if $x + y = x + z$, show that $y = z$.

1.3. Multiplication of natural numbers. Like addition, the multiplication of natural numbers is also defined recursively.

Definition 2.14 (Multiplication, $\cdot$)**.** The multiplication of natural numbers is defined recursively as follows

$$a \cdot 0 = 0$$
$$a \cdot S(b) = a + (a \cdot b)\,.$$

Exercise 2.15. Show that $1 = S(0)$ is the multiplicative identity. That is, for all a, we have

$$1 \cdot a = a \cdot 1 = a\,.$$

Exercise 2.16. Show that multiplication is commutative, and associative.

Exercise 2.17. Show that multiplication distributes over addition, namely

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)\,.$$

Remark 2.18. In fancy terms, the natural numbers $\mathbb{N}$, with the operations of addition and multiplication, form a *semiring*.

1.4. Ordering of natural numbers. With the successor function, we can define an ordering on $\mathbb{N}$s, as follow.

Definition 2.19. Let $n, m \in \mathbb{N}$. We say that $n \leq m$ iff there exists $k \in \mathbb{N}$ such that $m = n + k$.

We say $n < m$ iff $n \leq m$ and $n \neq m$.

Exercise 2.20. Prove that for all n, m in $\mathbb{N}$ exactly one of the following statements holds: i) $n < m$, $n = m$ or $n > m$.

1.5. A set-theory model for $\mathbb{N}$. Now we have an axiomatic description for $\mathbb{N}$, including the addition and the multiplication. But how do we know such a thing as $\mathbb{N}$ actually exists? Maybe there is some contradiction within the axioms which is only revealed after much investigation.

If we accept the axioms of ZF set theory, then the Peano axioms become a theorme. We can exhibit a set which satisfies all of these axioms; this will prove the axioms are consistent and give something sensible.

Definition 2.21 (Set-theoretic model for $\mathbb{N}$)**.** The von Neumann model of $\mathbb{N}$ is defined as the smallest set containing the element $0 := \{\ \} = \emptyset$, and closed under the successor function $S(n) := n \cup \{ n \}$.

Remark 2.22. The existence of such a set $\mathbb{N}$ is guaranteed by the Axiom of Infinity in Zamelo-Frankel (ZF) set theory. The axiom says there exists a set I, such that

$$\emptyset \in I \quad \text{and} \quad \forall x \in I, x \cup \{ x \} \in I\,.$$

Remark 2.23. With this model of set theory, the number $3 = S(S(S(0)))$ is really the following set

$$0 = \emptyset$$
$$1 = 0 \cup \{ 0 \} = \emptyset \cup \{ \emptyset \} = \{ \emptyset \}$$

$$2 = 1 \cup \{\,1\,\} = \{\,\emptyset\,\} \cup \{\,\{\,\emptyset\,\}\,\} = \{\,\emptyset, \{\,\emptyset\,\}\,\}$$
$$3 = 2 \cup \{\,2\,\} = \{\,\emptyset, \{\,\emptyset\,\}\,\} \cup \{\,\{\,\emptyset, \{\,\emptyset\,\}\,\}\,\} = \{\,\underbrace{\emptyset}_{0}, \underbrace{\{\,\emptyset\,\}}_{1}, \underbrace{\{\,\emptyset, \{\,\emptyset\,\}\,\}}_{2}\,\}\,,$$

et cetera.

2. The Grothendieck group to define $\mathbb{Z}$

Lecture 2
25/10/2016

$\mathbb{N}$ is all well, and good. It contains the familiar 'counting' numbers $0, 1, 2, 3, \ldots$. We can add, and we can multiply, what more do you want? Subtraction and negative numbers would be useful: historically you would want this in order to describe debts and credits in financial transactions. Generally, being able to manipulate equations without unnecessary restrictions would appeal to a mathematician. But trying to subtract two natural numbers $5 - 12$ forces us outside the set of natural numbers. We need new numbers.

To make $\mathbb{Z}$ from $\mathbb{N}$, we need to add negative elements. How does one do this formally? We don't know what $-n$ means at the moment, but we would want it to represent the 'difference' between 0 and n. Or between 5 and $n + 5$. So we want to consider pairs (a, b) where $a - b$ is constant. Or rather, rearrange this to say two pairs (a, b) and (c, d) are equal when $a+d = b+c$, (keeping in mind $(a, b) \rightsquigarrow' a - b'$.

Definition 2.24. f Consider the equivalence relation $\sim$ defined on $\mathbb{N} \times \mathbb{N}$ by

$$(a, b) \sim (c, d) \text{ if and only if } a + d = b + c\,.$$

The set of integers is defined as

$$\mathbb{Z} \coloneqq \mathbb{N} \times \mathbb{N}/\sim$$

Exercise 2.25. Check that $\sim$ is indeed an equivalence relation. That is, check

i) $\sim$ is reflexive: $x \sim x$,
ii) $\sim$ is symmetric: $x \sim y$ implies $y \sim x$, and
iii) $\sim$ is transitive: $x \sim y$ and $y \sim z$ implies $x \sim z$.

Since we know how to add in $\mathbb{N}$, we can define an addition on $\mathbb{Z}$ in a natural way.

Definition 2.26. The addition in $\mathbb{Z}$ is defined by

$$[(a, b)] + [(c, d)] = [(a + c, b + d)]\,.$$

Exercise 2.27. Check that this addition on $\mathbb{Z}$ is well defined. Show that $[(0, 0)]$ is the additive identity for addition on $\mathbb{Z}$. Show that the additive inverse of $[(a, b)]$ is $[(b, a)]$.

Now we have a formal construction of $\mathbb{Z}$, we can actually prove things about the integers.

Theorem 2.28. *Addition of integers is commutative, that is*

$$a + b = b + a$$

for any integers $a, b \in \mathbb{Z}$.

PROOF. Choose representatives $a = (\alpha, \beta)$ and $b = (\gamma, \delta)$. Then

$$a+b = [(\alpha,\beta)]+[(\gamma,\delta)] = [(\alpha+\gamma,\beta+\delta)] = [(\gamma+\alpha,\delta+\beta)] = [(\gamma,\delta)]+[(\alpha,\beta)] = b+a\,,$$

where we have used that addition in $\mathbb{N}$ is commutative. □

Exercise 2.29. Show that addition of integers is associative.

From the above, we get

Theorem 2.30. *The integers $\mathbb{Z}$ form an abelian group under addition.*

Example 2.31. Some elements of $K_0(\mathbb{N})$, for example, include

$$[(5,2)] = [(6,3)] = [(120,117)] \rightsquigarrow 3\,.$$

But we also have

$$[(2,5)] = [(3,6)] = [(117,120)] \rightsquigarrow \text{`}{-3}\text{'}\,.$$

We can check that

$$[(2,5)] + [(5,2)] = [(7,7)] = [(0,0)] = 0_{\mathbb{Z}}\,,$$

so indeed $(2,5) = -3$.

2.1. Multiplication of integers. We know how to add integers, by reducing the addition to that of natural numbers. How should we multiply them?

Definition 2.32. The multiplication on $\mathbb{Z}$ is defined by

$$[(a,b)] \cdot [(c,d)] = [(ac+bd, ad+bc)]\,.$$

Think that $(a-b)(c-d) = (ac+bd)-(ad+bc)$

Theorem 2.33. *Multiplication of integers is commutative.*

PROOF. After choosing representatives, we have that

$$[(a,b)] \cdot [(c,d)] = [(ac+bd, ad+bc)] = [(ca+db, da+cb)] = [(c,d)] \cdot [(a,b)]$$

□

Exercise 2.34. Check that multiplication is associative. Show that $1 = [(1,0)]$ is the multiplicative identity. Show that multiplication distributes over addition. (Hence $\mathbb{Z}$ is a commutative ring.)

Exercise 2.35. Show that $\mathbb{Z}$ is an integral domain. That is, if $ab = 0$, then $a = 0$ or $b = 0$.

2.2. Ordering of integers. We can view $\mathbb{N} \hookrightarrow \mathbb{Z}$ via $f\colon n \mapsto [(n,0)]$. We can use this to define an ordering on $\mathbb{Z}$.

Definition 2.36. We call $\mathbb{Z}_+ \coloneqq f(\mathbb{N} \setminus \{\, 0 \,\}) \subset \mathbb{Z}$ the positive integers.

If $a, b \in \mathbb{Z}$, we say

$$a < b \iff b - a \in \mathbb{Z}_{\geq 0}\,.$$

Exercise 2.37. Let $a \in \mathbb{Z}$. Prove that exactly one of the following statements holds: i) $a \in \mathbb{Z}_+$, ii) $a = 0$, or iii) $-a \in \mathbb{Z}_+$.

Exercise 2.38. Check that $<$ on $\mathbb{Z}$ is consistent with $<$ on $\mathbb{N}$. That is for $n, m \in \mathbb{N}$, show that $m < n$ in $\mathbb{N}$, if and only if $f(m) < f(n)$ in $\mathbb{Z}$.

2.3. The Grothendeick group in general. The above construction can be applied to any commutative monoid (set with commutative addition, and identity element).

Definition 2.39 (Grothendeick group)**.** Let M be a commutative monoid. The Grothendeick group $K_0(M)$ is defined by

$$K_0(M) = M \times M/\sim \ ,$$

where

$$(a,b) \sim (c,d) \text{ if and only if } a+d+m = b+c+m \text{ for some } m \text{ in } M\,.$$

(The extra $+m$ is necessary because the monoid may not be cancellative. It may not be true that $a+m=b+m$ implies $a=b$.)

The addition on $K_0(M)$ is defined by

$$[(m_1,n_1)] + [(m_2,n_2)] = [(m_1+m_2, n_1+n_2)]\,.$$

Theorem 2.40. *This addition makes $K_0(M)$ into a commutative group.*

One can find a homomorphic copy of M in $K_0(M)$ via $f\colon MtoK_0(M)$ with $f(m) = (m,0)$. But f is not necessarily injective, unless M is cancellative.

3. The field of fractions to define $\mathbb{Q}$

So now, we can add, subtract, and multiply. Do we need anything else? How to divide integers? Trying to divide two integers $3 \div 2$ forces us outside the set of integers, so we need some more numbers.

The construction of $\mathbb{Q}$ from $\mathbb{Z}$ has a very similar flavour to the construction of $\mathbb{Z}$ from $\mathbb{N}$. This is called the 'field of fractions' of an integral domain.

Definition 2.41. Let R be a integral domain. Then the field of fractions $F = \operatorname{Frac}(R)$ is the smallest field containing R. More concretely, we can define

$$\operatorname{Frac}(R) = \{\ (a,b) \in R \times R \setminus \{\ 0\ \}\ \} / \sim \ ,$$

where $\sim$ is the equivalence relation defined by $(a,b) \sim (c,d)$ iff $ad = bc$.

We should think that $(a,b) \leftrightsquigarrow \frac{a}{b}$ in this construction.

Definition 2.42. In the case $\mathbb{Z}$, we define $\mathbb{Q} \coloneqq \operatorname{Frac}(\mathbb{Z})$.

Exercise 2.43. How should addition and multiplication be defined in $\mathbb{Q}$? More generally how should addition and multiplication be defined in $\operatorname{Frac}(R)$? Remember that we want (a,b) to behave like $\frac{a}{b}$. Show that this makes $\operatorname{Frac}(R)$ into a field.

Exercise 2.44. Show that $g\colon R \to \operatorname{Frac}(R)$, $g(r) = [(r,1)] \leftrightsquigarrow \frac{r}{1}$ gives an embedding of $R \hookrightarrow \operatorname{Frac}(R)$. (Specifically, check that the operations are compatible, and the map is injective.)

Exercise 2.45. Set $\mathbb{Q}_+ \coloneqq \{\ [(a,b)] \mid a,b \in \mathbb{Z}_+ \text{ or } a,b \in -\mathbb{Z}_+\ \}$. For $p,q \in \mathbb{Q}$, define $p < q$ iff $q - p \in \mathbb{Q}_+$. Show that this agrees with $<$ for $\mathbb{Z}$.

The ordering on $\mathbb{Q}$ is *Archimedean*. This means roughly that there are no infinitely large/infinite small elements. More precisely for any element $x \in \mathbb{Q}$, there exists a natural number $n \in \mathbb{N}$, such that $x < n$.

4. Metric completition to form $\mathbb{R}$

So now we can add, subtract, multiply, and we can divide by non-zero numbers. Isn't this good enough? Well for lots of purposes yet, since this means $\mathbb{Q}$ is a field. But when you start doing calculus and analysis you run into problems with limits. Even before then, Euclidean geometry forces you away from $\mathbb{Q}$.

We know by Pythagoras that the diagonal of a unit square has length $\sqrt{2}$. We also that $\sqrt{2} \notin \mathbb{Q}$.

From analysis we have that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \,.$$

All the terms of the sequence $a_n = (1 + 1/n)^n$ are rational numbers. But the limit e is irrational.

This means that $\mathbb{Q}$ is not complete, it has 'holes' at all of the irrational points. We need to fill them to make $\mathbb{R}$. We say that $\mathbb{R}$ is the metric completion of $\mathbb{Q}$.

Definition 2.46 (Dedekind cut)**.** A *Dedekind cut* cut is an ordered pair (A, B) of subsets of $\mathbb{Q}$, satisfying

i) A and B are non-empty
ii) A and B are complements in $\mathbb{Q}$
iii) $a < b$ for all $a \in A$ and all $b \in B$.

Definition 2.47. A cut (A, B) is *normalised* if B does not contain a minimal element. If (A, B) is not normalised, then the *normalisation* is $(\widehat{A}, \widehat{B})$,where

- $\widehat{A} = A \cup \min(B)$ and $\widehat{B} = B \setminus \min B$.

Definition 2.48. A *real number* is a normalised cut. The set of all real numbers is denoted $\mathbb{R}$. (Notice here there is no equivalence, different cuts are different real numbers.)

Call a real number (A, B) rational if A contains a maximal element. Otherwise (A, B) is irrational.

Example 2.49. The cut $(A, B) = (\{\, x \in \mathbb{Q} \mid x < 0 \text{ or } x^2 < 2 \,\}, \{\, x \in \mathbb{Q} \mid x > 0, x^2 > 2 \,\})$ is not rational. This cut corresponds to $\sqrt{2}$.

Exercise 2.50. Define $h\colon \mathbb{Q} \to \mathbb{R}$ by

$$h(q) = (A_q, B_q)\,,$$

where

$$A_q \coloneqq \{\, x \in \mathbb{Q} \mid x \leq q \,\}\,, B_q \coloneqq \{\, x \in \mathbb{Q} \mid x > q \,\}\,.$$

Check that h is an embedding of $\mathbb{Q}$ into $\mathbb{R}$. (That is, it is an injective map.) Check also that h is a bijection $\mathbb{Q}$ to rational real numbers.

Definition 2.51. Define $0_{\mathbb{R}} = h(0)$, and $1_{\mathbb{R}} = h(1)$.

4.1. Arithmetic in $\mathbb{R}$. We want to define $+$ and $\cdot$ on $\mathbb{R}$. But, like in the previous consructions, they should somehow be compatible with the operations in $\mathbb{Q}$. The definitions are slightly more delicate this time.

Definition 2.52. Let $(A_1, B_1), (A_2, B_2)$ be cuts. Define

$$(A_1, B_1) + (A_2, B_3) := (\widehat{A_3}, \widehat{B_3}),$$

where

$$A_3 := \{ x \in \mathbb{Q} \mid \exists a_1 \in A_1, a_2 \in A_2 \text{ such that } x \leq a_1 + a_2 \},$$

and

$$B_3 = \mathbb{Q} - A_3 .$$

Exercise 2.53. Check that (A_3, B_3) above is a cut. Give an example where (A_3, B_3) is not already normalised.

Definition 2.54. Call a cut (A, B) negative if $0 \in B$, non-negative if $0 \in A$, and positive if A contains some $q \in \mathbb{Q}_+$.

Exercise 2.55. Suppose (A, B) is a cut. Show that $(-B, -A)$ is a cut, where $-A = \{ -a \mid a \in A \}$.

Show that $(\widehat{A}, \widehat{B})$ is positive iff $(\widehat{-B}, \widehat{-A})$ is negative.

Definition 2.56. Let $x = (A_1, B_1)$, $y = (A_2, B_2)$ be non-negative real numbers. Define

$$x \cdot y = (\widehat{A_3}, \widehat{B_3}),$$

where

$$A_3 := \{ x \in \mathbb{Q} \mid \exists a_1 \in A_1, a_2 \in A_2, a_1 \geq 0, a_2 \geq 0, \text{ such that } x \leq a_1 a_2 \} .$$

If x or y is negative, use the identities

$$x \cdot y = -(x \cdot (-y)) = -((-x) \cdot y) = (-x) \cdot (-y),$$

to convert to the positive case, and apply this definition.

Exercise 2.57. Check that (A_3, B_3) above is a cut.

Exercise 2.58. How should y^{-1} be defined in terms of cuts?

Exercise 2.59. Check that $g \colon \mathbb{Q} \to \mathbb{R}$ is an embedding of fields. That is, check that $\mathbb{R}$ is a field and that g preserves the operations.

Exercise 2.60. Let $\mathbb{R}_+$ be the set of all positive real numbers. Define $r < s$ iff $s - r \in \mathbb{R}_+$. Check that this is compatible $<$ on $\mathbb{Q}$ in the sense that $p < q$ iff $g(p) < g(q)$.

If $x = (A_1, B_1)$ and $y = (A_2, B_2)$ are real numbers, prove that $x < y$ iff $A_1 \subset A_2$.

4.2. Completeness of $\mathbb{R}$, the least upper bound property. The big payoff from this construction is that fact that $\mathbb{R}$ is now a complete metric space, it has no holes or gaps.

Theorem 2.61 (Least upper bound property)**.** *Every bounded set $S \subset \mathbb{R}$ has a least upper bound.*

Equivalently every Cauchy sequence $\{ a_n \}_n$ in $\mathbb{R}$ converges to some limit $L = \lim_{n\to\infty} a_n$. (Where Cauchy means eventually the terms of the sequence become arbitrarily close: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $n, m > N$ implies $|a_n - a_m| < \varepsilon$.

5. Algebraic completion to make $\mathbb{C}$

So now we can add, subtract, multiply, divide by non-zero elements, and take limits, all the while saying inside $\mathbb{R}$. Do we have enough numbers now? Not if we want to do algebra. Trying to solve certain polynomial equations forces one to consider square roots of negative numbers, even to get the real solutions.

Applying the cubic formula to solve $x^3 - 15x - 4 = 0$ leads to the computation of

$$\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}\,,$$

even to get the real root $x = 4$. (Bombelli, L'algebra, 1572)

It turns out that $\sqrt[3]{2 \pm \sqrt{-121}} = 2 \pm i$, so we can find the real root but we have to go through the complex numbers.

Definition 2.62. We define $\mathbb{C}$ to be the set

$$\mathbb{C} := \{\, (a, b) \mid a, b \in \mathbb{R} \,\}\ ,$$

with the addition

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)\,,$$

and multiplication

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1)\,.$$

In this definition think that $i \leftrightsquigarrow (0, 1)$.

Exercise 2.63. Check that $\mathbb{C}$, with these operations, forms a field. What is the additive identity? The multiplicative identity? What are multiplicative inverses?

Exercise 2.64. Define the map $k\colon \mathbb{R} \to \mathbb{C}$ by $k(r) = (r, 0)$. Check that this is an embedding of fields.

Now we can start to write $(a, b) \in \mathbb{C}$ as $a+bi$. We can also think of $\mathbb{C}$ as the quotient ring $\mathbb{R}[i]/(i^2 + 1)$, so that we work with polynomials in some symbol i, and have the rule $i^2 = -1$.

5.1. Algebraic completeness.

Much like with $\mathbb{R}$, there is a big payoff to be made from constructing $\mathbb{C}$.

Theorem 2.65 (Fundamental Theorem of Algebra)**.** *The complex numbers form an algebraically closed field. That is, every non-zero polynomial $f(z)$ with complex coefficients has at least one complex root.*

Equivalently, every non-constant polynomial $f(z)$ with complex coefficients and degree n has exactly n roots counted with multiplicity.

Finally do we have enough numbers? No, this is just the beginning. There are many points along the way from

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

where we can branch off in a new direction, and construct some new type of number.

We can branch off between $\mathbb{N}$ and $\mathbb{Z}$, and construct the *ordinal numbers*. Or between $\mathbb{Q}$ and $\mathbb{R}$, we can branch off and construct the *p-adic* numbers.

Between $\mathbb{R}$ and $\mathbb{C}$, instead of adjoining i with $i^2 = -1$ to get the complex numbers, we can adjoin j with $j^2 = 1$ to get the split complex numbers. Or maybe adjoin ϵ with $\epsilon^2 = 0$, and get the dual numbers.

Or we can keep repeating the (Cayley-Dickson) construction which took us from $\mathbb{R}$ to $\mathbb{C}$, and build the Hamiltonian numbers $\mathbb{H}$ from $\mathbb{C}$, and the octionion numbers $\mathbb{O}$ from $\mathbb{H}$, and the Sedenion numbers $\mathbb{S}$ from $\mathbb{O}$, and so on...

Or perhaps we clear all of this away, and build a completely new number system from scratch, using some of these ideas (Peano axioms, and Dedekind cuts) as inspiration. We will do this to build the surreal numbers **No**.

CHAPTER 3

The Surreal numbers No - Introduction

The surreal numbers form an incredibly rich number system, despite having such a simple definition. Constrast the natural numbers and various extensions with the surreal numbers. The natural numbers are defined using Peano axioms, then built on top of multiple times via 'complicated' definitions to form the integers, rationals, and finally reals. Whereas the surreal numbers start with two simple axioms, plus one rule for each of addition and multiplication. And already the surreal numbers contain and supersede everything up to the reals. Lecture 3 02/11/2016

1. Conway's axioms for the surreal numbers

Before giving the axioms/definitions of a surreal number, you should try to forget everything you know about numbers. You don't know what + means, what $\leq$ means, etc. You don't know what 0 is, ...

Axiom 3.1 (Conway, 1975)**.**

- A *surreal number* x is a pair $\{ X_L \mid X_R \}$, where $X_L = \left\{ x^L \right\}$ and $X_R = \left\{ x^R \right\}$ are sets of (previously created) surreal numbers, and no member if X_R is $\leq$ any member of X_L. Any surreal number arises in this way.

  In terms of logical symbols this means $\forall x^L \in X_L$, $\forall x^R \in X_R : \neg x^R \leq x^L$. Equivalently $\neg(\exists x^L \in X_L,\ \exists x^R \in X_R : x^R \leq x^L)$.

We call X_L the *left* set, and X_R the *right* set. If x is a surreal number, then the left set will be (capital $x)_L$ and the right set (capital $x)_R$. The elements of X_L are x^L, and the elements of X_R are x^R.

Of course, the first question is what does $x \leq y$ mean for surreal numbers? We need this to make sense of the first axiom.

Axiom 3.1 (ctd)**.**

- Let $x = \{ X_L \mid X_R \}$ and $y = \{ Y_L \mid Y_R \}$ be two surreal numbers. We say $x \leq y$ ("x is *less than or equal* to y") if and only if $y \leq$ no x^L and no $y^R \leq x$.
  Symbolically $x \leq y$ iff $\forall x^L \in X_L : \neg(y \leq x^L)$ and $\forall y^R \in Y_R : \neg(y^R \leq x)$.
  Equivalently $x \leq y$ iff $\neg(\exists x^L \in X_L : y \leq x^L)$ and $\neg(\exists y_R \in Y_R : y^R \leq x)$.

Notation-Definition 3.2. We say $x \geq y$ iff $y \leq x$. We say $x \not\leq y$ iff $\neg(x \leq y)$, that is iff $x \leq y$ does not hold.

How wonderfully peculiar these axioms are. Two inter-defined and recursively-defined axioms. To know what a surreal number is we need to nkow what a surreal number is, and what less than or equal to is. And to know what less than or equal to is, we need to know what less than or equal to is.

Remark 3.3. Conway's axioms originally read

- A surreal number is $\{ X_L \mid X_R \}$, where no x_L is $\geq$ any x_R.
- We say $x \geq y$ iff no $x^R \leq y$ and $x \leq$ no y^L.

With the defined equivalence $x \leq y$ iff $y \geq x$, these axioms are equivalent to the ones we give.

It might also be easier to remember if we write the definition of $\leq$ as

- $x \leq y$ iff no $x^L \geq y$ and $x \geq$ no y^R.

Remark 3.4. By dropping the requirement that $\{ X_L \mid X_R \}$ has no $x^R \leq$ any x^L, we obtain Conway's more general notion of a *game*. Many of the results can be generalised from surreal numbers to games.

We can also define some more shorthand for relations $\not\leq, \not\geq, <, >, =$, also.

Notation-Definition 3.5. We say $x \not\leq y$ iff $\neg(x \leq y)$, that is iff $x \leq y$ does not hold.

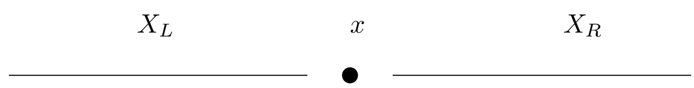
We say $x = y$ iff $x \leq y$ and $y \leq x$. We say $x < y$ iff $x \leq y$ and $y \not\leq x$. We say $y > x$ iff $x < y$.

Warning 3.6. Bear in mind that we have *defined* what it means for two surreal numbers to be equal. We therefore distinguish between the 'value' of a surreal number, and the 'form' of a surreal number. It might be helpful to have a notion that two surreal numbers are identical (have exactly the same form, same sets at every level) by writing $x \equiv y$, which means every $x^L \equiv$ some y^L and every $x^R \equiv$ some y^R.

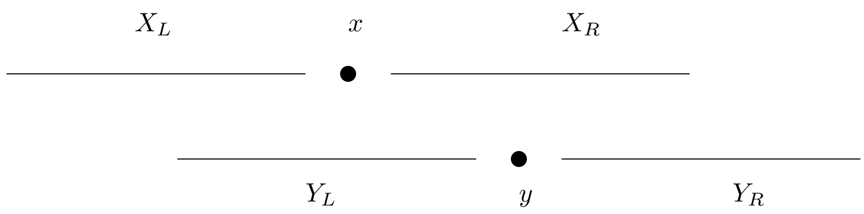
Notice also how we have defined $x \leq y$, $y \leq x$, etc. From $x \not\leq y$ we cannot conclude (at the moment) that $y \leq x$. This will be a result we have to prove.

Some motivations for these axioms are in order.

Motivation 3.7. Conway wants $x = \{ X_L \mid X_R \}$ to be between all numbers x^L (to the left) and all numbers x^R (to the right).



This will be proven later, and will mean that $\leq$, $+$ (when we define it), etc, have all the usual properties: $\leq$ should be transitive, total, etc. If we want x to lie between all x^L and all x^R, what should we demand for $x \leq y$ to behave as we would want?



The definition of $x \leq y$ says no $x^L \geq y$ and $x \geq$ no y^R.

Suppose that some $x_1^L \geq y$. Then we could not allow it to be the case that $x \leq y$. Otherwise we would have $x \leq y \leq x_1^L$, meaning $x \leq x_1^L$ by transitivity. So for $x \leq y$, we require no $x^L \geq y$.

Similarly, if $x \geq$ some y_1^R. Then we cannot allow $x \leq y$, for we would have $y_1^R \leq x \leq y$, and by transitivity $y_1^R \leq y$. So for $x \leq y$, we require $x \geq$ no y^R.

This is the spirit of Conway's definitions generally. "What do we know already, by the answers to simpler questions, about the object being defined." We can see this in action when we define addition, negation, multiplication, ...

2. The first surreal number (Day 0)

We have a wonderful circular definition of surreal numbers in terms of $\leq$ and other surreal numbers, which themselves are defined in terms of $\leq$ and other surreal numbers. Where can we possibly begin?

Even without knowing any surreal numbers, we can still form *sets* of surreal numbers. Namely the empty set $\emptyset$. So let us try

$$z \equiv \{ \, \emptyset \mid \emptyset \, \}$$

Notation 3.8. We will avoid writing $\emptyset$ unless necessary. If $X_L = A \cup B \cup \{ \, y \, \}$, we will write $\{ \, A, B, y \mid \cdots \}$ for simplicity.

So we can write $z \equiv \{ \ \mid \ \}$.

Is z a number? For this we need to check that no member of Z_R is $\leq$ any member of Z_L. Symbolically:

$$\forall z^L \in Z_L \, \forall z^R \in Z_R : z^R \not\leq z^L \, .$$

Since $Z_L = Z_R = \emptyset$, such a statement is vacuously true. If it were false, then we would have

$$\exists z_1^L \in Z_L \, \exists z_1^R \in Z_R : z_1^R \leq z_1^L \, .$$

In particular $\exists z_1^L \in Z_L$, and $\exists z_l^R \in Z_R$, so $Z_L \neq \emptyset$ and $Z_R \neq \emptyset$.

So z is a number.

Is $z \leq z$? For this we need to check that no $z^L \geq z$, and $z \geq$ no z^R. Symbolically

$$\forall z^L \in Z_L : z^L \not\geq z \text{ and } \forall z^R \in Z_R : z \not\geq z^R \, .$$

Since $Z_L = \emptyset$, and $Z_R = \emptyset$, this is vacuously true. If it were false, then we would have

$$\exists z_1^L \in Z_L : z_1^L \geq z \text{ or } \exists z_1^R \in Z_R : z \geq z_!^R \, ,$$

in particular $\exists z_1^L \in Z_L$ so $Z_L \neq \emptyset$, or $\exists z_1^R \in Z_R$ so $Z_R \neq \emptyset$.

So $z \leq z$. And by the definition of $=$, we also obtain $z = z$.

Remark 3.9. It will turn out that $z + x = x$ and $zx = z$, for any number x, when we define multiplication of surreal numbers. So perhaps a good name for z is 0?

3. The next surreal numbers (Day 1)

Now we have a surreal number z, we can use it in the left or the right set to make new surreal numbers. What choices do we have for X_L? $X_L = \emptyset$, or $X_L = \{\, z \,\}$. Similarly $X_R = \emptyset$, or $X_R = \{\, z \,\}$.

Taking all combinations of choices leads to the following collection of potential new surreal numbers.

$$\{\ \mid\ \}, \{\, z \mid\ \}, \{\ \mid z \,\}, \{\, z \mid z \,\}\,.$$

We already dealt with $\{\ \mid\ \} \equiv z$. So let's look at the other ones.

Is $\{\, z \mid\ \}$ a number? To check if $x = \{\, z \mid\ \}$ is a number, we need to check that no $x^R \leq$ any x^L. Symbolically

$$\forall x^L \in X_L\, \forall x^R \in X_R : x^R \not\leq x^L\,.$$

Since $X_R = \emptyset$, this holds vacuously. If it were false, then

$$\exists x^L \in X_L\, \exists x^R \in X_R : x^R \leq x^L\,,$$

in particular $\exists x^R \in X_R$ so $X_R \neq \emptyset$.

Thus $\{\, z \mid\ \}$ is a number.

Is $\{\ \mid z \,\}$ a number? To check if $x = \{\ \mid z \,\}$ is a number, we need to check that no $x^R \leq$ any x^L. Symbolically

$$\forall x^L \in X_L\, \forall x^R \in X_R : x^R \not\leq x^L\,.$$

Since $X_L = \emptyset$, this holds vacuously. If it were false, then

$$\exists x^L \in X_L\, \exists x^R \in X_R : x^R \leq x^L\,,$$

in particular $\exists x^L \in X_L$ so $X_L \neq \emptyset$.

Thus $\{\ \mid z \,\}$ is a number.

Let us write $p \equiv \{\, z \mid\ \}$, and $m \equiv \{\ \mid z \,\}$, as names for these numbers.

Obviously, we can generalise both of these result to the following.

Proposition 3.10. *If $X_L = \emptyset$, or $X_R = \emptyset$. Then*

$$\{\, X_L \mid X_R \,\}$$

is a surreal number.

PROOF. We need to check that no $x^R \leq$ any x^L. Symbolically

$$\forall x^L \in X_L\, \forall x^R \in X_R : x^R \not\leq x^L\,.$$

Since one of $X_L = \emptyset$ or $X_R = \emptyset$, this is vacuously true.

If it were false, then

$$\exists x^L \in X_L\, \exists x^R \in X_R : x^R \leq x^L\,,$$

in particular $\exists x^L \in X_L$ so $X_L \neq \emptyset$ and $\exists x^R \in X_R$ so $X_R \neq \emptyset$. □

Is $\{ z \mid z \}$ a number? What about $x \equiv \{ z \mid z \}$? Is this a number? We need to check that no $x^R \leq$ any x^L. Symbolically

$$\forall x^L \in X_L \, \forall x^R \in X^R : x^R \not\leq x^L \,.$$

By taking $x^R = z \in X_R$ and $x^L = z \in X_L$, we have $z \leq z$. So this fails. And $\{ z \mid z \}$ is not a number.

[It is a game, under Conway's more general definition. As a game, it is often called $* \equiv \{ 0 \mid 0 \}$. It corresponds to a game where the first player to move loses (where loses means the player has no valid moves). We can return to this later.]

What can we prove about p, m and z now?

Is $z \leq p$? $z \leq p$ means that no $z^L \geq p$, and $z \geq$ no p^R. Symbolically

$$\forall z^L \in Z_L : z^L \not\geq p \text{ and } \forall p^R \in P_R : z \not\geq p^R \,.$$

Since $Z_L = P_R = \emptyset$, this is true. (If it were false, then

$$\exists z^L \in Z_L : z^L \geq p \text{ or } \exists p^R in P_R : z \geq p^R \,.$$

In particualr $Z_L \neq \emptyset$ or $P_R \neq \emptyset$.)

So $z \leq p$.

Is $p \leq z$? $p \leq z$ means that no $p^L \geq z$, and $p \geq$ no z^L. Formally

$$\forall p^L \in P_L : p^L \not\geq z \text{ and } \forall z^L \in Z^L : p \not\geq z^L \,.$$

Since $Z_L = \emptyset$, the second part of the conjunction holds. But what about the first? Since $P_L = \{ z \}$, we need to check that $p^L = 0 \not\geq 0$. But from previous, we know $0 \geq 0$, and the first part of the conjunction fails.

This means $p \not\leq z$. And by the definition of $<$, we conclude $z < p$.

Is $m \leq z$? $m \leq z$ means that no $m^L \geq z$, and $m \geq$ no z^L. Formally

$$\forall m^L \in M_L : m^L \not\geq z \text{ and } \forall z^L \in Z^L : m \not\geq z^L \,.$$

Since $M_L = Z_L = \emptyset$ this is vacuously true. (If it were false, then

$$\exists m_1^L \in M_L : m_1^L \geq z \text{ or } \exists z^L in Z^L : m \geq z_1^L \,,$$

so $M_L \neq \emptyset$ or $Z_L \neq \emptyset$.)

Is $z \leq m$? $z \leq m$ means that no $z^L \geq m$ and $z \geq$ no m^L. Formally

$$\forall z^L in Z_L : z^l \not\geq m \text{ and } \forall m^L \in M^L : z \not\geq m^L \,.$$

Since $Z_L = \emptyset$, the first part of this conjunction is true. But when we check the second: $M_L = \{ z \}$, so we need to check that $z \not\geq m_1^L \equiv z$. But from previous, we know $z \geq z$. So the second part of the conjunction fails.

Hence $z \not\leq m$. By the definition of $<$, we obtain that $m < z$.

Is $m \le p$? Well, if $\le$ behaves like we would want it to, then yes, this must hold because $m \le 0 \le p$, so by transitivity we should have $m \le p$. But we haven't proven transitivity yet, so we must check this directly.

So $m \le p$ means no $m^L \ge p$ and $m \ge$ no p^R. Formally

$$\forall m^L in M_L : m^L \not\ge p \text{ and } \forall p^R \in P_R : m \not\ge p^R \,.$$

Since $M_L = \emptyset$ and $P_R = \emptyset$, this is vacuously true.

We can generalise this as follows.

Proposition 3.11. *Let $x = \{ \mid X_R \}$ and $y = \{ Y_L \mid \}$ be surreal numbers. Then $x \le y$. Moreover $x \le 0$ and $0 \le y$.*

PROOF. To show $x \le y$, we check that no $x^L \ge y$, and $x \ge$ no y^R. Formally

$$\forall x^L \in X_L : x^L \not\ge y \text{ and } \forall y^R \in Y_R : x \not\ge y^R \,.$$

Since $X_L = \emptyset$ and $Y_R = \emptyset$ both parts of the disjunction are true. Hence $x \le y$.

Moreover, taking $Y_L = \emptyset$ gives $x \le \{ \mid \} \equiv 0$. Alternatively, taking $X_R = \emptyset$ gives $0 \equiv \{ \mid \} \le y$. □

Is $p \le m$? This is true if no $p^L \ge m$ and $p \ge$ no m^R. But $0 \in P_L = \{ z \}$, and we have $z \ge m$, so the first condition fails. Therefore $p \not\le m$.

Since $m \le p$ and $p \not\le m$, we conclude $p < m$.

Is $m \le m$? This is true if no $m^L \ge m$, and $m \ge$ no m^R. Since $M_L = \emptyset$, the first condition holds vacuously. (If the first condition fails, then there is some m^L_1 with $m \le m^L_1$, in particular $\left\{ m^L_1 \right\} \subset M_L \ne \emptyset$.) Since $M_R = \{ z \}$, and $m \not\ge z$, the second condition holds. Therefore $m \le m$. We also get $m = m$ too.

Is $p \le p$? This is true if no $p^L \ge p$ and $p \ge$ no p^L. Since $P_R = \emptyset$ the second condition is true vacuously. (If it were false, then there is some p^L_1 with $p \ge p^L_1$, in particular $\left\{ p^L_1 \right\} \subset p^L \ne \emptyset$.) Since $P_L = \{ z \}$, and $z \not\ge p$, the first condition holds. Therefore $p \le p$. We also have $p = p$ too.

This means we have now three numbers

$$m < z < p \,.$$

And we have proven the various equailities and inequailities. It will turn out later that $px = x$ for any number x, when we have defined multiplication. So perhaps we should name p as 1. It also turns out to be the case that $m = -p$, so let's call m as -1. With the name 0 for z, we therefore get the following.

$$-1 < 0 < 1 \,.$$

4. Numbers created on Day 2

We can now use any of the following 8 sets as the left and right sets. How many combinations give *valid* numbers?

$$\emptyset, \{ 0 \}, \{ 1 \}, \{ -1 \}$$
$$\{ 0, 1 \}, \{ -1, 0 \}, \{ -1, 1 \}, \{ -1, 0, 1 \}$$

With a bit of work, one can find the following 20 valid surreal numbers formed from the above sets.

$$\begin{gathered}\{\,|\,\},\quad \{\,0\,|\,\},\quad \{\,-1\,|\,\},\quad \{\,1\,|\,\},\quad \{\,-1,0\,|\,\},\\ \{\,0,1\,|\,\},\quad \{\,-1,1\,|\,\},\quad \{\,-1,0,1\,|\,\},\quad \{\,-1\,|\,0\,\},\quad \{\,|\,0\,\},\\ \{\,|\,-1\,\},\quad \{\,|\,1\,\},\quad \{\,-1\,|\,1\,\},\quad \{\,0\,|\,1\,\},\quad \{\,-1,0\,|\,1\,\}\\ \{\,|\,0,-1\,\},\quad \{\,|\,0,1\,\},\quad \{\,-1\,|\,0,1\,\},\quad \{\,|\,-1,1\,\},\quad \{\,|\,-1,0,1\,\}\,.\end{gathered}$$

Exercise 3.12. Check this! Make sure we haven't missed any.

Lecture 4
09/11/2016

Some of these numbers we have already encountered. We've got $0 \equiv \{\,|\,\}$, $1 \equiv \{\,0\,|\,\}$ and $-1 \equiv \{\,|\,0\,\}$ on the list again. So there are still potentially 17 new numbers to deal with.

We can check directly how any pair of these numbers (new or old) compares. It's going to be tedious to check every pair, but in principle we could do it. For example

Result $\{\,1\,|\,\} > 1$: We check $\{\,1\,|\,\} \geq 1$, and $\{\,1\,|\,\} \not\leq 1$.

For the first: $1 \leq \{\,1\,|\,\}$ means no $1^L \geq \{\,1\,|\,\}$ and $1 \geq$ no $\{\,1\,|\,\}^R$. Since $\{\,1\,|\,\}^R = \emptyset$, the second part holds vacuously. For the first we need to check that $0 \not\geq \{\,1\,|\,\}$.

Subresult $0 \not\geq \{\,1\,|\,\}$: We have $\{\,1\,|\,\} \leq 0$ means no $\{\,1\,|\,\}^L \geq 0$ and $\{\,1\,|\,\} \geq$ no 0^R. The first condition fails because $1 \in \{\,1\,|\,\}^L = \{\,1\,\}$ has $1 \geq 0$. So $\{\,1\,|\,\} \not\leq 0$, or equivalently $0 \not\geq \{\,1\,|\,\}$.

Since the condition $0 \not\geq \{\,1\,|\,\}$ holds, we see the first condition no $1^L \geq \{\,1\,|\,\}$ holds. Since both conditions holds, we get that $\{\,1\,|\,\} \geq 1$.

So we check that $\{\,1\,|\,\} \not\leq 1$. Now $\{\,1\,|\,\} \leq 1$ means no $\{\,1\,|\,\}^L \geq 1$ and $\{\,1\,|\,\} \geq$ no 1^R. For the first $1 \in \{\,1\,|\,\}^L = \{\,1\,\}$ has $1 \geq 1$, so the first condition fails. Thus $\{\,1\,|\,\} \not\leq 1$.

Summary: $\{\,1\,|\,\} > 1$.

Notice how the recursive nature of $\leq$ made its way into the proof.

Result $0 < \{\,0\,|\,1\,\}$ and $\{\,0\,|\,1\,\} < 1$. . For the first part, we check $0 \leq \{\,0\,|\,1\,\}$ and $0 \not\geq \{\,0\,|\,1\,\}$.

Now $0 \leq \{\,0\,|\,1\,\}$ means no $0^L \geq \{\,0\,|\,1\,\}$ (vacuously true), and $0 \geq$ no $\{\,0\,|\,1\,\}^R = \{\,1\,\}$ (holds because $0 \not\geq 1$).

Now $\{\,0\,|\,1\,\} \leq 0$ means no $\{\,0\,|\,1\,\}^L \geq 0$ (fails because $0 \in \{\,0\,|\,1\,\}^L = \{\,0\,\}$ has $0 \geq 0$) and .... So $\{\,0\,|\,1\,\} \not\leq 0$.

Summary $0 < \{\,0\,|\,1\,\}$.

Exercise 3.13. Complete the proof to show $\{\,0\,|\,1\,\} < 1$.

More interesting is the following

Result $\{ -1 \mid 1 \} = 0$: We check $\{ -1 \mid 1 \} \leq 0$: this means no $\{ -1 \mid 1 \}^L \geq 0$ (indeed $-1 \not\geq 0$), and $\{ -1 \mid 1 \} \geq$ no 0^R (indeed $0^R = \emptyset$).

Also we check $0 \leq \{ -1 \mid 1 \}$: this means no $0^L \geq \{ -1 \mid 1 \}$ (indeed $0^L = \emptyset$), and $0 \geq$ no $\{ -1 \mid 1 \}^R$ (indeed $0 \not\geq 1$).

So on the second day, we have a 'new' number which equals one of our old numbers, even though it has a different form. Notice $\{ -1 \mid 1 \} \not\equiv 0$, because they are formed by different sets. But these two representations have the same value 0.

I'm going to give names to some of these numbers now, even though we will only establish later the properties (say, of addition $+$), which will fully justify the names.

Let's define

$$
\begin{aligned}
2 &\equiv \{ 1 \mid \ \} \\
\frac{1}{2} &\equiv \{ 0 \mid 1 \} \\
-\frac{1}{2} &\equiv \{ -1 \mid 0 \} \\
-2 &\equiv \{ \ \mid -1 \} .
\end{aligned}
$$

I claim that the following equalities hold

$$
\begin{aligned}
2 &= \{ 1 \mid \ \} = \{ 0, 1 \mid \ \} = \{ -1, 1 \mid \ \} = \{ -1, 0, 1 \mid \ \} \\
1 &= \{ 0 \mid \ \} = \{ -1, 0 \mid \ \} \\
\frac{1}{2} &= \{ 0 \mid 1 \} = \{ -1, 0 \mid 1 \} \\
0 &= \{ \ \mid \ \} = \{ -1 \mid \ \} = \{ \ \mid 1 \} = \{ -1 \mid 1 \} \\
-\frac{1}{2} &= \{ -1 \mid 0 \} = \{ -1 \mid 0, 1 \} \\
-1 &= \{ \ \mid 0 \} = \{ \ \mid 0, 1 \} \\
-2 &= \{ \ \mid -1 \} = \{ \ \mid -1, 0 \} = \{ \ \mid -1, 0 \} = \{ \ \mid -1, 0, 1 \} .
\end{aligned}
$$

I also claim that the following inequalities hold for any representative of these numbers

$$
-2 < -1 < -\frac{1}{2} < 0 < \frac{1}{2} < 1 < 2 ,
$$

including any which should follow by transitivity. And that if $x < y$ holds then $x \not\geq y$, etc. All of these results can be checked explicitly using the definitions.

Exercise 3.14. Check some of these directly. Check, say, $\{ -1 \mid 0 \} < \{ -1 \mid 1 \}$. You *may not* use that $\{ -1 \mid 1 \} = 0$ to say $-1 \not\geq \{ -1 \mid 1 \}$ since we have not yet proved that $x = y$ and $y \not\geq z$ implies $x \not\geq z$!

None of the proofs are any more difficult (other than keeping track of the nesting). Any of these proofs will necessarily terminate a condition that holds vacuously because all of the steps are *simpler.* Recall

$$
x \leq y \iff \text{no } x^L \geq y \text{ and } x \geq \text{no } y^R .
$$

In the first part, x is replaced with x^L which is constructed earlier. Similarly in the second part, y is replaced with y^R, which is constructed earlier.

Let's also consider some of the patterns in these equalities. The 'interpretation' is that $\{ X_L \mid X_R \}$ should be between all values in X_L and all values in X_R. Consider $x = \{ -1 \mid 0 \}$, this should have $x > -1$ and $x < 0$. Consider also $y = \{ -1 \mid 0, 1 \}$ this should have $y > -1$, and $y < 0$ and $y < 1$. But the condition $y < 0$ should imply that $y < 1$ anyway, so the extra 1 is irrelevant. And indeed it is, since we have $\{ -1 \mid 0 \} = \{ -1 \mid 0, 1 \}$.

We would like be able to prove a result like the following:

Conjecture 3.15. *If $x = \{ X_L \mid X_R \}$, and $y \not\leq x$. Then $x = \{ X_L \mid X_R, y \}$.*

If $x = \{ X_L \mid X_R \}$, and $y \not\geq x$. Then $x = \{ X_L, y \mid X_R \}$.

We can do this, but we have to prove some general results about surreal numbers first. So we must first talk about how to prove results for all surreal numbers.

5. Proof by induction for surreal numbers

Recall the first of Conway's axioms defining surreal numbers:

- A *surreal number* x is a pair $\{ X_L \mid X_R \}$, where $X_L = \left\{ x^L \right\}$ and $X_R = \left\{ x^R \right\}$ are sets of (previously created) surreal numbers, and no member if X_R is $\leq$ any member of X_L. Any surreal number arises in this way.

The final part of this is what we want to focus on. We can formalise this notion by way of a type of 'descending chain condition'. Knuth's characters refer to this as 'no infinite ancestral chains'.

Axiom 3.16 (Descending chain condition)**.** There is no infinite sequences of numbers $x_i = \{ X_L^i \mid X_R^i \}$ with $x_{i+1} \in X_L^i \cup X_R^i$ for all $i \in \mathbb{N}$.

[Compare this with the axiom of regularity in ZF set theory, which reads

$$\forall x (x \neq \emptyset \implies \exists y \in x : y \cap x = \emptyset)\,.$$

That is, every non-empty A contains an element which is disjoint from A.

This means that no set is a member of itself. For if $x = \{ x \}$, then the element $x \in x = A$ is not disjoint from $x = A$.

It also means that there are no downward infinite chains a_n, such that

$$a_{i+1} \in a_i \text{ for all } i.$$

]

From this we can establish Conway induction for surreal numbers.

Theorem 3.17 (Conway induction)**.** *Let P be a property which surreal numbers can have, and suppose that $\{ X_L \mid X_R \}$ has the property P whenever all elements of X_L and X_R have the property. Then all surreal numbers has this property.*

PROOF. Suppose x_0 is a surreal number which does not satisfy property P. If all of the elements of X_L and X_R satisfy P, then so does X. So there is some number $x_1 \in X_L \cup X_R$ which does not satisfy P.

Now suppose x_n does not satisfy P. As before we can find $x_{n+1} \in X_L^n \cup X_R^n$ which does not satisfy P.

By induction we can find x_n which does not satisfy P for all $n \in \mathbb{N}$, with $x_{n+1} \in X_L^n \cup X_R^n$.

Thus there is an infinite chain of surreal numbers, contrary to the descending chain condition. Therefore there is no surreal number which does not satisfy P. $\square$

Alternatively, you might want to take Conway induction as an axiom of the construction of real numbers. Then we can prove the descending chain condition using the property $P(x)$ "there is no chain of numbers $x_0, x_1, x_2, \ldots$, with every $x_{i+1} \in X_L^i \cup X_R^i$".

We may therefore use Conway induction to prove a certain statement hold for all surreal numbers.

6. Properties of $\leq$

Firstly, it will be necessary to use the concept of a game in some of the later proofs. We also might want to compare/contrast between the properties of games, and the properties of numbers.

Axiom 3.18 (Game)**.**
- A *game* $g = \{ X_L \mid X_R \}$ is a pair of sets X_L and X_R of (previously created) games.
- All games arise in this way, i.e. Conway Induction works for games too.

Notice, every surreal number is a game, since we don't even need to check the condition no $x^L \geq$ any x^R. We can start off the construction of games with $0 = \{ \mid \}$, also. But the thing $\{ 0 \mid 0 \}$ that we threw away on day 1, is a valid game now.

These are games in the following sense. The game $g = \{ X_L \mid X_R \}$ is between two players, left and right. The elements of X_L indicate which positions left can move to from the current position, and the elements of X_R indicate which positions right can move to from the current position. A game ends when the player to move has no valid moves. E.g. if $g = \{ \mid X_R \}$, and left is about to move, then the game ends, and he loses (by convention).

(E.g. tic-tac-toe, with some tweaks to avoid draws...)

Then we have

- $G > 0$ if there is a winning strategy for left
- $G < 0$ if there is a winning strategy for right
- $G = 0$ if there is a winning strategy for player 2
- $G \parallel 0$ if there is a winning strategy for player 1

E.g. respectively $1 = \{ 0 \mid \}$, $-1 = \{ \mid 0 \}$, $0 = \{ \mid \}$, and $* = \{ 0 \mid 0 \}$.

Let's use this idea of induction to prove some results about $\leq$ on surreal numbers.

Earlier on we claimed the interpretation that $x = \{ X_L \mid X_R \}$ lies between all elements of X_L, and all elements of X_R. This can indeed be seen in various examples above, though perhaps rather trivially in some cases. We will work towards this

Firstly, we have

Theorem 3.19. *For all surreal numbers x, the following holds:*

$x \leq x$. As a consequence we have $x = x$.

PROOF. This will be proven by (Conway) induction. We start with the base case, and indeed $0 \leq 0$.

Take as surreal number $x = \{ X_L \mid X_R \}$. Assume this theorem holds for all of x's parents. We want to show now that $x \leq x$.

By definition, this means we need to check that no $x^L \geq x$, and $x \geq$ no x^R.

Applying the definition of $\leq$ to $x \leq x^L$ (some fixed x^L) means we need to check that no $(x^L)' \geq x^L$ and $x \geq$ no $(x^L)^R$. But the first condition fails by taking $(x^L)' = x^L \in X_L$. (Induction assumption, the theorem is true for all previous cases, including x^L.) Therefore $x \nleq x^L$, and hence $x^L \ngeq x$.

Similarly, we check $x^R \leq x$ (some fixed x^R). But by definition, this means no $(x^R)^L \geq x$, and $x^R \geq$ no $(x^R)'$. But the second condition fails by taking $(x^R)' = x^R \in X_R$. (Induction assumption.) Hence $x^R \nleq x$, or equivalently $x \ngeq x^R$.

So we have checked by definition that $x \leq x$. The theorem is proven by induction. And $x = x$ follows immediately by the definition of $=$. □

Notice here that we did not use that x is a (well-formed) surreal number. So this result is true for games too.

We can also show that $\geq$ is transitive

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Theorem 3.20. *If $x \leq y$, and $y \leq z$, then $x \leq z$.*

PROOF. Suppose this is true for any parents of x, y, z. To prove the claim by induction we check the definition of $x \leq z$.

We need to check no $x^L \geq z$, and $x \geq$ no z^L.

If we have some $x^L \geq z$, then from $y \leq z$, we obtain $y \leq x^L$ using the induction assumption. But $x \leq y$ means no $x^L \geq y$. Hence no $x^L \geq z$.

If we have $x \geq$ some z^L, then we obtain from $x \leq y$, that $z^L \leq y$ using the induction assumption. But $y \leq z$ means $y \geq$ no z^L. Hence $x \geq$ no z^L.

So by definition of $\leq$, we get $x \leq z$.

(Do we need to check a base case? What is the base case? We drop one option at a time, so we will eventually get back to a case where $Z_L = \emptyset$ and $X_L = \emptyset$. Then the condition $x \geq$ no z^L holds vacuously, as does the condition no $x^L \geq z$.

The base case of induction is $x = \{ \mid X_R \}$ and $z = \{ Z_L \mid \}$. In this case we trivially have $x \leq z$, so in particular we have $x \leq y$ and $y \leq z$ implies $x \leq z$.) □

We can now use this to show that surreal numbers are totally ordered, and that $x = \{ X_L \mid X_R \}$ lies between all the left and right elements.

Theorem 3.21. *For any number x, we have $x^L < x < x^R$. Moreover, for any wo surreal numbers x, y, we must have $x \leq y$ or $y \leq x$.*

Proof. As part of the proof that $x \leq x$, we established that $x^L \ngeq x$, and $x \ngeq x^R$. So we only need to establish now that $x^L \leq x$ and $x \leq x^R$.

But $x^L \leq x$ means no $(x^L)^L \geq x$ and $x^L \geq$ no x^R. The second condition is part of the definition of a number, so holds. Also we cannot have $(x^L)^L \geq x$, otherwise by the induction assumption we get $(x^L)^L < x^L$, so that $x \leq x^L$, a contradiction with our earlier result. So $x^L \leq x$.

Similarly $x \leq x^R$ means no $x^L \geq x^R$, and $x \geq$ no $(x^R)^R$. The first is the definition of a number. We cannot have $x \geq (x^R)^R$, otherwise by the induction assumption we get $x^R < (x^R)^R$, so that $x^R \leq x$, contradicting an earlier result. So $x \leq x^R$.

Now if it is not the case that $x \leq y$, then we have $x \nleq y$. So one of the conditions no $x^L \geq y$, or $x \geq$ no y^R fails.

If the former, then some $x^L \geq y$, and so $x > x^L \geq y$ meaning $x \geq y$. If the latter, then $x \geq$ some y^R, nd so $x \geq y^R > y$ meaning $x \geq y$.

(The base case is, of course, when $x^L = X^R = \emptyset$, and we have vacuously that $x^L < 0 < x^R$, because there are no elements in X_L or X_R which make this false. □

So surreal numbers are totally ordered. Notice here that we did use the well-formedness of a surreal number in the proof. This suggests that this theorem could fail for games. Indeed it does.

We do, now, have the consequence that

$$x \nleq y \implies y < x\,.$$

This is because $x \nleq y$ implies $y \leq x$, which taken together with $y \ngeq x$ gives by definition $y < x$.

Exercise 3.22. Check (using the definitions exactly as for numbers) that $0 \nleq \{\,0 \mid 0\,\}$, and $\{\,0 \mid 0\,\} \nleq 0$. So games 0 and $\{\,0 \mid 0\,\}$ cannot be compared. $\{\,0 \mid 0\,\}$ is said to be 'fuzzy' against 0.

Now we can go back and prove our earlier 'conjecture'.

Theorem 3.23. *If $x = \{\,X_L \mid X_R\,\}$, and $y \ngeq x$, then $a \equiv \{\,y, X_L \mid X_R\,\}$ is equal to x. If $y \nleq x$, then $b \equiv \{\,X_L \mid X_R, y\,\}$ is equal to x.*

Proof. We check that $a \leq x$. This requires no $a^L \geq x$, and $a \geq$ no x^R.

Every a^L is either y, or an x^L. We have $y \ngeq x$ by assumption. We have $x^L < x$, meaning $x^L \ngeq x$. Since a is a number (check!), we know that $a < a^R = x^R$, so $a \ngeq$ any x^R. So indeed $a \leq x$.

We check also that $x \leq a$. This requires no $x^L \geq a$, and $x \geq$ no a^R.

Every a^R is an x^R, and since $x < x^R$, we indeed have $x \ngeq x^R = a^R$. Similarly, $a^L < a$, and since every x^L is an a^L, we have $x^L < a$ meaning $x^L \ngeq a$.

So $a = x$ □

Exercise 3.24. Complete the proof for the $y \not\leq x$, and $b \equiv \{ X_L \mid y, X_R \}$ has $b = x$ case.

Exercise 3.25. Use this theorem to justify the equalities given between the day 2 numbers. Convince yourself that only the largest element of X_L and the smallest element of X_R matter. Use the previous theorem to justify the inequalities between the day 2 numbers.

How many numbers are there on day 3?

We'll see shortly, that we can write $n + 1 = \{ n \mid \}$, rather like the construction of the natural numbers in set theory.

With this theorem we can now consider some more fanciful numbers.

$$\{ 0, 1, 2, 3, \ldots \mid \}$$

must be $> n$ for any n. Therefore this is infinite in size. Let's call it ω.

Then what about

$$\{ 0, 1, 2, 3, \ldots \mid \omega \} .$$

This must be $> n$ for any n. But also $< \omega$. Therefore this is an infinitely large number, which is less than infinity. It turns out that this really have value $\omega - 1$, in the sense that $(\omega - 1) + 1 = \omega$. We will need to define addition to justify this though. . .

Also we have (whatever the fractions means. . .)

$$\epsilon = \left\{ 0 \,\middle|\, 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\} .$$

This must be > 0, and $<$ any $\frac{1}{n}$. It must be infinitesimal. It turns out that $\epsilon = \frac{1}{\omega}$.

Let's start to justify these names, by introducing the arithmetic of surreal numbers.

CHAPTER 4

Arithmetic of surreal numbers

We've spent a lot of time talking about surreal numbers, and the order properties of $\leq$. We assigned temporary names to various numbers, with the promise that we would justify the names later.

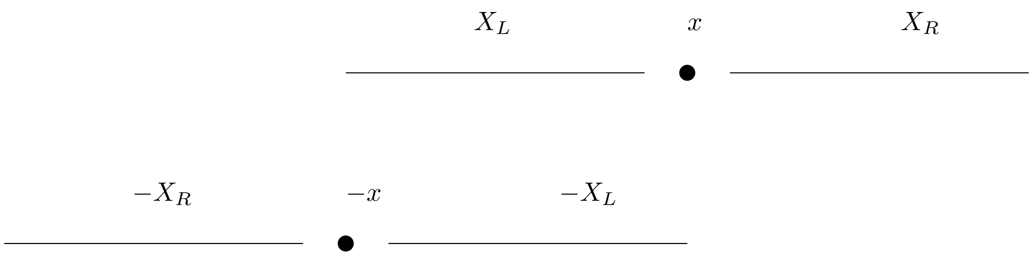
1. Negation

Firstly, we give the definition of negation of surreal numbers

Definition 4.1. Let $x \equiv \{ X_L \mid X_R \}$ be a surreal number. Then

$$-x \equiv \{ -X_R \mid -X_L \} ,$$

where $-A = \{ -a \mid a \in A \}$.



If $\leq$ behaves, then we should have $x^L \leq x$ giving $-x \leq -x^L$, so the right set of $-x$ consists of all $-x^L$'s.

Example 4.2. Check that $-2, -1, -\frac{1}{2}$ match with this definition.

We need to check that $-x$ is a number.

Proposition 4.3. *Let $x = \{ X_L \mid X_R \}$ be a number. Then $-x$ is a number.*

Proof. This is by Conway induction. (Base case says that $-0 = \{ \mid \}$ is a number. True vacuously.)

We know that $-x^L$ and $-x^R$ are numbers. We want to check that $-x_L \not\geq -x_R$.

We will inductively prove that $a \leq b$ iff $-b \leq -a$.

$a \leq b$ means no $a^L \geq b$ and $a \geq$ no b^R. And $-b \leq -a$ means no $(-b)^L \geq -a$ and $-b \geq$ no $(-a)^R$.

We have now what

$$-b \geq (-a)^R \underbrace{\iff}_{(-A)_R = -(A_L)} -b \geq -a^L \underbrace{\iff}_{\text{induction}} a^L \geq b \, ,$$

similarly

$$(-b)^L \geq -a \underbrace{\Longrightarrow}_{(-b)_L = B_R} -b^R \geq -a \underbrace{\Longleftrightarrow}_{\text{induction}} a \geq b^R \,.$$

So returning to the original equation, we have that $x_R \not\geq x_L$, since x is a number. We conclude that $-x_L \not\geq -x_R$. Therefore $-x$ is a number. □

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From this, we get results like $x \geq 0$ implies $-x \leq 0$.

We can also prove that $-(-x) \equiv x$.

Proposition 4.4. *For any number x, we have $-(-x) \equiv x$.*

Proof. This is an example of what Conway calls a 1-line proof. It works using induction, and just manipulates the definition.

$$-(-x) \equiv \left\{ -(-x)^R \;\middle|\; -(-x)^L \right\} \equiv \left\{ -(-x^L) \;\middle|\; -(-x^R) \right\} = \left\{ x^L \;\middle|\; x^R \right\} \equiv x \,.$$

Inside this one line, we have used the induction assumption that $-(-x)$ holds for all parents of x. □

Now let us move on to addition.

2. Addition

We define addition on surreal numbers as follows

Definition 4.5. Let $x = \{ X_L \mid X_R \}$ and $y = \{ Y_L \mid Y_R \}$ be surreal numbers. Then

$$x + y \equiv \left\{ x^L + y, x + y^L \;\middle|\; x^R + y, x + y^R \right\} \,.$$

This definition is motivated by considering $x^L < x < x^R$, so we should have $x^L + y < x + y < x^R + y$. Similarly $y^L < y < y^R$, so $x + y^L < x + y < x + y^R$. So we know what should go in the left and right sets.

Exercise 4.6. We also have that $x^L + y^L < x + y < x^R + y^R$. But this is weaker than the two inequalities above, so the following operation is probably not the same as $x + y$. Investigate this operation

$$x \oplus y = \left\{ x^L \oplus y^L \;\middle|\; x^R \oplus y^R \right\} \,.$$

We should check that $x + y$ is a number. But to do this, we need to establish some further properties. For the moment, we take $x + y$ to be a definition of $+$ on games.

We show later than if x and y are both numbers, then so is $x + y$.

We can now start checking whether some of the names we assigned earlier are justified.

We compute

$$1 + 1 = \left\{ 1^L + 1, 1 + 1^L \;\middle|\; \underbrace{1^R + 1, 1 + 1^R}_{1^R = \emptyset} \right\} = \{ 0 + 1, 1 + 0 \mid \}$$

We need to compute $0 + 1$ and $1 + 0$ now, but you can check directly that both are 1. SO

$$1 + 1 = \{ 1, 1 \mid \} = \{ 1 \mid \} \equiv 2 \,.$$

Theorem 4.7. *For any number x, we have*

$$x + 0 \equiv 0 + x \equiv x\,.$$

PROOF. This is another 1-line proof. We have

$$x + 0 \equiv \left\{ x^L + 0, x + 0^L \,\middle|\, x^R + 0, x + 0^R \right\} \equiv \left\{ x^L + 0 \,\middle|\, x^R + 0 \right\} \equiv \left\{ x^L \,\middle|\, x^R \right\} \equiv x\,.$$

Same for $0 + x$. □

So indeed 0 is the additive identity, 0 was a good name for the number $\{\ |\ \}$. We also see that 2 was a good name for $\{\,1\,|\ \}$.

Similarly, $\{\,2\,|\ \} = 3, \{\,3\,|\ \} = 4, \ldots$.

It is more difficult to check that

$$\frac{1}{2} + \frac{1}{2} = 1\,,$$

because we need more sub-results on surreal numbers. Computing the sum should lead to

$$\begin{aligned} \frac{1}{2} + \frac{1}{2} &= \left\{ \frac{1}{2}^L + \frac{1}{2}, \frac{1}{2} + \frac{1}{2}^L, \frac{1}{2}^R + \frac{1}{2}, \frac{1}{2} + \frac{1}{2}^R \right\} \\ &= \left\{ \frac{1}{2} \,\middle|\, 1\frac{1}{2} \right\}, \end{aligned}$$

where $1\frac{1}{2} \equiv \{\,1 \mid 2\,\} = 1 + \frac{1}{2}$.

So we know this is between $\frac{1}{2}$ and $1\frac{1}{2}$, but what number is it? Directly, you can check this is $= 1$, and it is easier using some of the results we proved above. But it would be nice to have some high-powered theorem which decides automatically it for us.

Example 4.8. We have $\frac{1}{2} + \frac{1}{2} = 1$.

We check $x \equiv \left\{ \frac{1}{2} \,\middle|\, 1\frac{1}{2} \right\} \leq 1$. This means no $x^L \geq 1$, and $x \geq$ no 1^R. First true since $\frac{1}{2} < 1$, i.e. $\ngeq$. Second is true since $1^R = \emptyset$.

We check $1 \leq x \equiv \left\{ \frac{1}{2} \,\middle|\, 1\frac{1}{2} \right\}$. This means no $1^L \geq x$, and $1 \geq$ no x^R. The second is true since $1 \geq \{\,1 \mid 2\,\}$ implies $1 \geq \{\,1 \mid 2\,\} > 1$, which is false. The first is true since $0 \geq x$ implies $0 \geq x > \frac{1}{2} > 0$, so $0 > 0$, which is false.

We can also check

$$(\omega - 1) + 1 = \{\,1, 2, 3, \ldots, \omega - 1 \mid \omega + 1\,\}\,.$$

We have $\omega = \{\,1, 2, 3, \ldots \mid\ \}$. Since $\omega - 1 \ngeq \omega$, we can insert this into the left set without changing the value. Since $\omega + 1 \nleq \omega$, we can add it to the right set without changing the value. Hence

$$(\omega - 1) + 1 = \omega\,.$$

Let's prove some more properties about +

Theorem 4.9. *Addition of surreal numbers is commutative, and associative.*

PROOF. These have one line proofs

$$x+y \equiv \left\{ x^L + y, x + y^L \;\middle|\; x^R + y, x + y^R \right\} \equiv \left\{ y + x^L, y^L + x \;\middle|\; y + x^R, y^R + x \right\} \equiv y+x\,.$$

$$\begin{aligned}
&(x+y)+z \equiv \left\{ (x+y)^L + z, (x+y) + z^L \;\middle|\; \cdots \right\} \\
&\equiv \left\{ (x^L+y)+z, (x+y^L)+z, (x+y)+z^L \;\middle|\; \cdots \right\} \\
&\equiv \left\{ x^L + (y+z), x + (y+z)^L \;\middle|\; \cdots \right\} \\
&\equiv \left\{ x^L + (y+z), x + (y^L+z), x + (y+z^L) \;\middle|\; \cdots \right\} \equiv x + (y+z)\,.
\end{aligned}$$

□

Fill in the missing details about the right hand set.

Slightly more interesting is the following that $-x + x = 0$.

Theorem 4.10. *For any number x, $-x + x = 0$.*

PROOF. Suppose that $0 \not\leq (-x + x)$, then there is some $0 \geq (-x + x)^R$ or $0^L \geq (-x + x)$ (no.).

But $0 \geq (-x+x)^R$ means $0 \geq (-x)^R + x$or $0 \geq -x + x^R$, which means $0 \geq -x^L + x$.

But by induction we have $x^R + (-x^R) \geq 0$, and $x^L + (-x^L) \geq 0$. So we see that $-x + x^R \leq 0$ is false: we would have to check $(-x + x^R)^L \not\geq 0$, but one $(-x + x^R)^L$ is $(-x)^L + x^R = -(x^R) + x^R \geq 0$. Similarly $-x^L + x \leq 0$ is false: we would have to check $(-x^L + x)^L \not\geq 0$, but one $(-x^L + x)^L$ is $-x^L + x^L \geq$.

Therefore, we do have $-x + x \geq 0$. Similarly $-x + x \leq 0$. So they are equal. □

Notice here that we only have $x + (-x) = 0$, and not $\equiv 0$. E.g. working out $1 + (-1) = \{\, 0 \mid \;\} + \{\; \mid 0 \,\} = \{\, 0 + (-1) \mid 1 + 0 \,\} = \{\, -1 \mid 1 \,\}$, and this is $= 0$, but not $\equiv 0$.

Remark 4.11. We see that under $+$, surreal numbers form an abelian semigroup when we even demand $\equiv$. Moreover, under $=$, they form an abelian group. (We actualy should call this a GROUP, not a group since the surreal numbers are so large they form a CLASS not a set.)

We can also check the following

Exercise 4.12. For all numbers x, y, we have

$$-(x+y) \equiv -x + -y\,.$$

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2.1. Addition, and order. We now establish some properties about $+$, with respect to $\leq$. This will allow us to show that $x + y$ is a binary operation on surreal numbers, i.e. the sum of two numbers is still a number.

Theorem 4.13. *For any (games) x, y, z, we have $y \leq z$ iff $y + x \leq z + x$.*

PROOF. We show $y + x \leq z + x \implies y \leq z$, and $y + x \not\leq z + x \implies y \not\leq z$.

Result 1: If $y+x \leq z+x$, we have that no $(y+x)^l \geq z+x$, and $y+x \geq$ no $(z+x)^R$. This means no $y + x^L \geq z + x$, no $y^L + x \geq z + x$, $y + x \geq$ no $z + x^R$, and $y + x \geq$ no $z^R + x$. The important ones here are the second, and fourth. By the induction hypothesis, these imply no $y^L \geq z$, and $y \geq$ no z^R. This by definition means $y \leq z$.

Result 2: Suppose $y + x \not\leq z + x$, and $y \leq z$. From the first, we have that some

$$y + x^L \geq z + x, y^L + x \geq z + x, y + x \geq z^R + x, \text{ or } y + x \geq z + x^R .$$

By the induction hypothesis and transitivity (which holds for games), using $y \leq z$, we get some

$$z + x^L \geq z + x, y^L + x \geq y + x, z + x \geq z^R + x, \text{ or } z + x \geq z + x^R .$$

And using the induction hypothesis again, we get

$$x^L \geq x, y^L \geq y, z \geq z^R, \text{ or } x \geq x^R ,$$

all of which are false. □

Remark 4.14. From this, we see that $y = z$ implies $y + x = z + x$, so that addition of surreal numbers (games at the moment) is well defined, when using $=$ instead of $\equiv$.

Theorem 4.15. *If x and y are surreal numbers, then so if $x + y$.*

PROOF. Since x is a number, we have $x^L < x < x^R$, and so we deduce $x^L + y < x + y < x^R + y$. Similarly $y^L < y < y^R$, so $x + y^L < x + y < x + y^R$.

By induction, we have that $x^L + y, x + y^L, x^R + y, x + y^R$ are all numbers, and that $x^L + y, x + y^L < x + y < x + y^R, x^R + y$. This means, no elements of $(x + y)^L$ are $\geq$ any elements of $(x + y)^R$.

Thus we deduce that $x + y$ is also a surreal number. □

2.2. The numbers on day n. In problem sheet 3, you are guided through a proof by induction that there are 2^n numbers on day n. If the numbers on day n are

$$x_1 < x_2 < \cdots < x_m ,$$

then on day $n + 1$, we obtain exactly

$$\{ \mid x_1 \} < x_1 < \{ x_1 \mid x_2 \} < x_2 < \cdots < \{ x_{m-1} \mid x_m \} < x_m < \{ x_m \mid \} .$$

That is we create new numbers at the ends, and in between the existing numbers. All that is left now is to settle the values of these numbers.

In the final part of the problem sheet, you will establish the following.

Theorem 4.16. *If x_m is the largest number on day n, then the largest number created on day $n + 1$ is $\{ x_m \mid \} = x_m + 1$.*

Theorem 4.17. *If a, b are surreal numbers, and there is no $a < w < b$, such that w is older (created earlier) than a, or is older than b. Then*

$$\{ a \mid b \} + \{ a \mid b \} = a + b .$$

PROOF. Exercise. □

That is to say,

$$\{\, a \mid b \,\} = \frac{1}{2}(a+b)\,.$$

We can now give the structure of surreal numbers on day n completely. After day n, we get the integers $-n, \dots, n$. And any dyadic rationals half way between all the numbers on the previous day.

Any dyadic rationals $\frac{n}{2^m}$ is created on some finite day. You can find it via a binary search algorithm: is it bigger, or smaller than the current number? Then restrict to the appropriate sub interval. (This leads to the notion of the more general sign expansion of a surreal number, later).

2.3. The remaining numbers. But what about the irrational numbers? Or even the fractions like $\frac{1}{3}$? When are they created? I've already indicated that we can discuss numbers where the left and right sets are infinite, to get $\omega = \{\, 0, 1, 2, \dots, \mid \,\}$. So we can try approximating it by dyadic rationals

$$\frac{1}{4} + \frac{1}{16} + \cdots + (\frac{1}{4})^n < \frac{1}{3} < \frac{1}{2} - \frac{1}{8} - \cdots - \frac{1}{2}(\frac{1}{4})^n\,.$$

Therefore, we might try

$$x = \{\, 1/4, 5/16, 21/64, \dots \mid 1/2, 3/8, 11/32, 43/128, \dots \,\}\ .$$

It can then be checked that $x + x + x = 3x = 1$, so we have found $\frac{1}{3}$. (Rather than something infinitesimally close.) For the proof, it is best to use the simplicity theorem (a version of which you should prove on the problem sheet)

Theorem 4.18 (Simplicity)**.** *Suppose for $x = \{\, X_L \mid X_R \,\}$ that some number z satisfies $x^L \not\geq z$ and $z \not\geq x^R$, for all x_L and x^R. But that no parent of z satisfies this. Then $x = z$.*

Proof. Exercise. □

[This says that $x = \{\, X_L \mid X_R \,\}$ is the simplest (i.e. earliest created) number lying between all X_L and all X_R. If z is the earliest such number, then it satisfies these properties. Its parents z^L and z^R cannot satisfy them, because they are created earlier. But the version above also holds for games x.]

Then we compute that

$$x + x = \left\{\, x + x^L \,\middle|\, x + x^R \,\right\}$$

$$x + x + x = \left\{\, x + x^L + (x^L)', x + x + x^L \,\middle|\, x + x^R + (x^R)', x + x + x^R \,\right\}$$

The numbers in the left hand set are < 1. Why? We know that $x <$ anything $> 1/3$, and that $x^L < 1/3$. If $x + x^L + (x^L)' \geq 1$, then $x + 1/3 + 1/3 > x + x^L + (x^L)' \geq 1$, so $x > 1/3$. Impossible. Similarly for the case. And the numbers in the right are > 1, since $x >$ anything $< 1/3$.

So now,

$$y = x + x + x = \{\, Y_L \mid Y_R \,\}\ ,$$

where all $y^L < 1$, and all $y^R > 1$. We claim that by the simplicity theorem $y = 1$. Indeed, $z = 1$ satisfies these conditions $y^L \not\geq 1$ and $1 \not\geq y^R$. But the parent $z^L = 0$

does not satisfy this, since the left set contains $y^L = x+1/4+1/4 > 1/4+1/4+1/4 = 3/4$, say. So $y = z = 1$.

You can find any other real number in a similar way.

2.4. Tree of surreal numbers. We can also draw a tree describing the surreal numbers on each day. Each node of the tree connects to two children: the first numbers created just to the left, and just to the right of the current number.

*** Insert picture of surreal number tree, see `https://en.wikipedia.org/wiki/Surreal_number#/media/File:Surreal_number_tree.svg` ***

3. Multiplication

Definition 4.19 (Multiplication)**.** Let $x = \{ X_L \mid X_R \}$ and $y = \{ Y_L \mid Y_R \}$ be numbers (or games). We define

$$xy \equiv \{x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R | \\ x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L\} .$$

This definition very much does need motivation.

We 'know' that if $x > x^L$, and $y > 0$, then $xy > x^L y$. But as a result, this is strictly weaker than the elements in the left set above. We instead use that

$$(x - x^L)(y - y^L) > 0 ,$$

giving

$$xy > xy^L + x^L y - x^L y^L .$$

Similarly for other elements of the left and right sets.

Exercise 4.20. For all x, y, z, we have the following identities and equalities

i) $x0 \equiv 0$,
ii) $x1 \equiv x$,
iii) $xy \equiv yx$,
iv) $(-x)y \equiv x(-y) \equiv -(xy)$,
v) $(x + y)z = xz + yz$,
vi) $(xy)z = x(yz)$

Why are the results in v) and vi) only =, and not ≡?

We again justify the name $0 \equiv \{ \mid \}$, since it behaves like 0 under multiplication. We also justify the name $1 \equiv \{ 0 \mid \}$, since it is the multiplicative identity.

But as with addition, we still need to prove that xy is a number. This requires an understanding of how ≤, multiplication and addition all relate. It ends up dealing with a rather intricate multiple step induction argument. Lecture 8 07/12/2016

Theorem 4.21.
- *If x and y are numbers, then so is xy,*
- *If x_1, x_2, y are numbers, and $x_1 = x_2$, then $x_1 y = x_2 y$,*
- *If x_1, x_2, y_1, y_2 are numbers, such that $x_1 \le x_2$, and $y_1 \le y_2$, then $x_1 y_2 + x_2 y_1 \le x_1 y_1 + x_2 y_2$. The conclusion is $<$ if both premises are $<$.*

PROOF OUTLINE. Call the inequality in iii) as $P(x_1, x_2 : y_1, y_2)$. Notice that for $x_1 \le x_2 \le x_3$, $P(x_1, x_3 : y_1, y_2)$ follows from $P(x_1, x_2 : y_1, y_2)$ and $P(x_2, x_3 : y_1, y_2)$.

To prove i), we can by induction assume that all parents of xy are numbers because they all have a form like $x^*y + xy^* - x^*y^*$. So we need to establish only that the result xy satisfies the no $(xy)^L \ge$ any $(xy)^R$ condition of surreal numbers.

This entails checking, 4 inequalities such as

$$\underbrace{x^{L_1}y + xy^L - x^{L_1}y^L}_{\text{first type of } (xy)^L} < \underbrace{x^{L_2}y + xy^R - x^{L_2}y^R}_{\text{first type of } (xy)^R} .$$

For this inequality, there are two cases to deal with $x^{L_1} \le x^{L_2}$, and $x^{L_1} \ge x_{L_2}$. In the first case we can write

$$\begin{aligned} x^{L_1}y + xy^L - x^{L_1}y^L &\le x^{L_2}y + xy^L - x^{L_2}y^L \\ &< x^{L_2}y + xy^R - x^{L_2}y^R \end{aligned}$$

using $P(x^{L_1}, x^{L_2} : y^L, y)$ for the first, and $P(x_2^L, x : y^L, y^R)$ for the second.

If $x^{L_2} \le x^{L_1}$ we can instead write

$$\begin{aligned} x^{L_1}y + xy^L - x^{L_1}y &< x^{L_1}y + xy^R - x^{L_1}y^R \\ &\le x^{L_2}y + xy^R - x^{L_2}y^R \end{aligned}$$

using $P(x^{L_1}, x : y^L, y^R)$ and $P(x^{L_2}, x^{L_1} : y, y^R)$ respectively.

For ii) we can verify that $x_1 = x_2$ iff $x_1^L < x_2 < x_1^R$ and $x_2^L < x_1 < x_2^R$ just using the definitions of $x_1 \le x_2$ and $x_2 \le x_1$. So we need to check various inequalities like $(x_1y)^L < x_2y < (x_1y)^R$.

We'll check the case $(x_1y)^L = x_1^Ly + x_1y^L - x_1^Ly^L < x_2y$. By induction we can assume $x_1y^L = x_2y^L$. We also have $x_1^Ly + x_2y^L < x_1^Ly^L + x_2y$ using $P(x_1^L, x_2 : y^L, y)$. Together they imply the required result.

For iii). If $x_1 = x_2$, or $y_1 = y_2$, ii) shows that $=$ holds, as we can assume $x_1 < x_2$ and $y_1 < y_2$. If $x_1 < x_2$, then either $x_1 < x_1^R \le x_2$ or $x_1 \le x_2^L < x_2$.

If $x_1 < x_1^R \le x_2$ holds, then we can deduce $P(x_1, x_2 : y_1, y_2)$ from $x_1, x_1^R : y_1, y_2)$ and $P(x_1^R, x_2 : y_1, y_2)$. The latter is strictly simpler than the original P. Using $y_1 < y_2$, the former can be reduced to one of $P(x_1, x_1^R : y_1, y_1^R)$ or $P(x_1, x_1^R : y_1^L, y_1)$.

Overall we reduce to $P(x^L, x : y^L, y), P(x^L, x : y, y^R), P(x, x^R : y^L, y), P(x, x^R : y, y^R)$. These claim that xy is between it's left and right parents, and so they can be reduced to strictly simpler P using i). □

Once you have proven all of the above exercise, you will have established that

Theorem 4.22. *Under $+, \times$ surreal numbers form a totally ordered, commutative RING.*

And, for example, we can conclude now that $xy = xz$ implies $y = z$, if $x \ne 0$. And if $x, y > 0$, then $xy > 0$. How?

Exercise 4.23. Above, we showed that multiplication is well-defined for surreal numbers up to $=$. It turns out that it is not well-defined for games. Find a game G_1 such that $\{ 1 \mid \ \} G_1 \ne \{ 0, 1 \mid \ \} G_1$, even though $\{ 0, 1 \mid \ \} = \{ 1 \mid \ \} = 2$.

4. Division

We now want to show that they in fact, form a FIELD. But how to find multiplicative inverses?

As a first, observation. There can't really be any formula like the ones for multiplication, and for addition. Any such formula to find $1/3 = \{\, 0 \mid \,\} / \{\, 2 \mid \,\}$, could only have finitely many elements in the left set, and in the right set. And so, would be a dyadic fraction. Hmm...

Thankfully, Conway has great insight about this question. We ill give a formula for y^{-1}, when $y > 0$. Then using properties of multiplication, we get $-(-y)^{-1}$ as the multiplicative inverse, when $y < 0$.

Firstly, we need an auxiliary result

Lemma 4.24. *Let $x > 0$ be a surreal number. Then x has a form in which $0 \in X_L$, and all other x^L are positive.*

Proof. Let $x = \{\, X_L \mid X_R \,\}$. Since $x > 0$, i.e. $x \not\leq 0$, it must be that some $x^L \geq 0$. Therefore, every $x^R > x^L \geq 0$. So putting

$$y = \{\, 0, X_L \mid X_R \,\}$$

definitely gives us a number. Moreover, since $0 \not\geq x$, we can insert 0 on the left, without changing the value, i.e. $x = y$.

Finally, consider

$$z = \left\{\, 0, \widetilde{X_L} \;\middle|\; X_R \,\right\} ,$$

where

$$\widetilde{X_L} = \left\{\, x^L \in X_L \mid x^L \geq 0 \,\right\} .$$

Then z is still, definitely, a number, and $z > 0$. If $x^L < 0$, then $x^L \not\geq z$, so adding it to the left set does not change the value. We conclude

$$z = y .$$

This produces the require form $\left\{\, 0, \widetilde{X_L} \;\middle|\; X_R \,\right\}$ for $x = \{\, X_L \mid X_R \,\}$. □

For a number x of this form, we are going to write x^L to mean the positive left elements. Conway, then gives the following remarkable definition of a number y, which we will prove is x^{-1}.

Definition 4.25 (Reciprocal)**.** Let $x > 0$ be given in the above form $x = \{\, 0, X_L \mid X_R \,\}$. Then define

$$y := \left\{\, 0, \frac{1 + (x^R - x)y^L}{x^R}, \frac{1 + (x^L - x)y^R}{x^L} \;\middle|\; \frac{1 + (x^L - x)y^L}{x^L}, \frac{1 + (x^R - x)y^R}{x^R} \,\right\} .$$

This definition most definitely requires explanation/motivation. As usual, it is a recursive definition, requiring us to know what $1/x^L$, and $1/x^R$ are already. But also there is a recursion going on with the elements of y itself. What does this mean?

Example 4.26. Let's apply this definition to $3 = \{ 2 \mid \}$. We write it in the form $\{ 0, 2 \mid \}$. There is no x^R, and the only positive x^L is 2. So we get that

$$y = \left\{ 0, \frac{1}{2}(1 - y^R) \;\middle|\; \frac{1}{2}(1 - y^L) \right\}.$$

We start with the first $y^L = 0$, to get $frac12(1 - 0) = \frac{1}{2}$ as a y^R. Feed this back in to get, $\frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}$ as a y^L. And then $\frac{1}{2}(1 - \frac{1}{4}) = \frac{3}{8}$ as a y^R. And so on.

We form

$$y = \left\{ 0, \frac{1}{4}, \frac{5}{16}, \dots \;\middle|\; \frac{1}{2}, \frac{3}{8}, \dots \right\},$$

which looks like the expression for $\frac{1}{3}$ that we found earlier.

We now establish that y is a number, and that $xy = 1$, showing that every non-zero element has a multiplicative inverse.

Theorem 4.27. *i) Firstly, we have $xy^L < 1 < xy^R$, for all y^L, y^R. And therefore y is a number.*
ii) We have $(xy)^L < 1 < (xy)^R$, for all $(xy)^L, (xy)^R$. And therefore, $xy = 1$.

Proof. For i) The parents of y are given by formulae of the form

$$y'' = \frac{1 + (x' - x)y'}{x'},$$

where y' is an earlier parent, and x' is some non-zero parent of x.

By induction $x'\frac{1}{x'} = 1$, so we can rewrite

$$y'' = \frac{1 + (x' - x)y'}{x'} = \frac{1}{x'} + \frac{x' - x}{x'}y'.$$

This can be written as

$$\begin{aligned} 1 - xy'' &= 1 - \frac{x}{x'} - \frac{x(x' - x)}{x'}y' \\ &= \frac{x' - x}{x'} - \frac{x(x' - x)}{x'}y' \\ &= (1 - xy')\frac{x' - x}{x'}. \end{aligned}$$

We see from this that if y' satisfies the condition, then so does y''.

For example with y'' being a new left parent, one case is

$$1 - xy'' = \underbrace{(1 - xy^R)}_{<0} \underbrace{\frac{x^L - x}{x^L}}_{<0} \geq 0,$$

so that $1 \geq xy''$.

By this induction over new parents, starting with $y^L = 0$ which has $0x = 0 < 1$, we see that all parents of y satisfy this condition.

Now since we have $xy^L < xy^R$, which conclude $y^L < y^R$, for all y^L, y^R. So there is no inequality of the form $y^L \geq y^R$. Hence y is a number.

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For ii) The parents of xy are of the form $x'y + xy' - x'y' = (1 + x(y - y''))$, for the above y''. From this, the condition holds.

For example, one of the left parents is $x^L y + xy^L - x^L y^L$, and we write this as

$$1 + \underbrace{x^L(y - \underbrace{\frac{1 + (x^L - x)y^L}{x^L}}_{\text{some } y^R})}_{<0} < 1\,.$$

Finally, we show that $xy = 1$. Write $z = xy$. Firstly, 0 is a left parent of z, by taking $x^L = y^L = 0$. And the above result claims that $z^L < 1 < z^R$, for all z^L, z^R.

We have $1 \leq z$. Otherwise some $1^L \geq z$ or $1 \geq$ some z^R. First doesn't hold since $0 < z$, by the previous, and the second doesn't hold since $1 < z^R$ by the above.

We have $z \leq 1$. otherwise some $z^L \geq 1$ or $z \geq$ some 1^R. Second doesn't hold since $1^R = \emptyset$. First doesn't hold since $z^L < 1$ by the previous.

(Alternatively, $z = 1$ satisfies the conditions, but 0 doesn't, as one $(xy)^L$ is 0. So $z = 1$ by simplicity.)

Thus $z = 1$. This shows that y is the multiplicative inverse of x. □

Conclusion: every non-zero surreal number has a multiplicative inverse, given by the previous definition in the positive case.

Therefore, surreal numbers in $+, \times$ form a totally ordered FIELD.

A more precise proof of the construction of $y = \frac{1}{x}$ is given in Section 3.4 of `https://arxiv.org/abs/math/0410026v2`

Exercise 4.28 (Challenge)**.** Try to find a similar construction for the square-root of a non-negative number x. I.e. give a definition for

$$y = \sqrt{x}$$

in terms of x^L, y^L, x^R, y^R, x.

Can you prove your claim?

CHAPTER 5

Real Numbers, Ordinal Numbers and Surreal Numbers

We now know that surreal numbers form a FIELD. Since $1 + 1 + \cdots + 1 > 0$, we know that this is a field of characteristic 0, and therefore contains $\mathbb{Q}$ as a subfield. How do the surreal numbers relate to other systems of numbers we know?

1. Real numbers

We can define a certain subset of surreal numbers, which we will call *real* numbers. We can then identify the usual real numbers (constructed via Dedekind cuts) with these Conway real numbers.

Definition 5.1 (Conway Real)**.** Let x be a surreal number. We call x *real* if and only if $-n < x < n$ (for some integer $n \in \mathbb{Z} \subset \text{Surreals}$) and

$$x = \left\{ x-1, x-\frac{1}{2}, x-\frac{1}{3}, \ldots, x-\frac{1}{n}, \ldots \;\middle|\; x+1, x+\frac{1}{2}, x+\frac{1}{3}, \ldots, x+\frac{1}{n}, \ldots \right\}.$$

It is useful here to remember the simplicity theorem

Theorem 5.2 (Simplicity)**.** *Suppose for $x = \{ X_L \mid X_R \}$ that some number z satisfies $x^L \ngeq z$ and $z \ngeq x^R$, for all x_L and x^R. But that no parent of z satisfies this. Then $x = z$. [If z is the simplest number between X_L and X_R, then $x = z$.]*

For example

- 0 is a Conway real: if
$$x = \left\{ -1, -\frac{1}{2}, \ldots \;\middle|\; 1, \frac{1}{2}, \ldots \right\}.$$
Every number in the left set is < 0, ever number in the right set is > 0. So $x^L < 0 < x^R$. Clearly no parent of 0 satisfies the same property (because 0 has no parents). So by the simplicity theorem $x = 0$.
- However $\epsilon = \left\{ 0 \mid 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\}$ is not real. If
$$x = \left\{ \epsilon-1, \epsilon-\frac{1}{2}, \ldots \;\middle|\; \epsilon+1, \epsilon+\frac{1}{2}, \ldots \right\}.$$
Then every element in the left set is < 0 because $\epsilon < \frac{1}{n}$ for any n. But, every element in the right set is > 0 because $\epsilon > 0$. So $x = 0$ as before.

By induction we have the following theorem

Theorem 5.3. *Let x be a rational number whose denominator divides 2^n. Then*
$$x = \left\{ x - \frac{1}{2^n} \;\middle|\; x + \frac{1}{2^n} \right\}.$$

From this we can see that all dyadic fractions are Conway real numbers.

Using the formulae for addition, multiplication, etc we can show

- If x is Conway real, then so is $-x$,
- If x, y are Conway real, then so is $x + y$,
- If x, y are Conway real, then so is xy.

The following theorem allows us to identiy Conway real numbers with the usual real numbers constructed viva Dedekind cuts.

Theorem 5.4. *i) Each Conway real hs a unique expression of the form* $\{ L \mid R \}$, *where* L, R *are non-empty sets of rationals,* L *has no greatest element,* R *has no least element, there is at most one rational not in* $L \cup R$, *and* L *is downwards closed* $(y < y' \in L \implies y \in L)$ R *is upwards closed.*

ii) Every such choice for L, R *as above gives a (unique) real number.*

We can therefore identify $\mathbb{R}$ with these Conway real numbers via

$$\begin{aligned} \mathbb{R} &\mapsto \text{Conway real} \\ r &\mapsto \{ \text{rationals} < r \mid \text{rationals} > r \} \end{aligned}$$

Of course, there is still some checking to be done to see that this is an embedding. But these are technical details.

Conway also gives a discussion of the 'best' order in which to construct the numbers up to $\mathbb{R}$, using some of the surreal number insight. See "The logical theory of real numbers" pages 25–27 of On Numbers and Games.

2. Ordinals

In the same way, we can construct an analogue of the ordinal numbers inside the surreals. Recall that ordinal numbers are (in one definition) equivalence classes of well-ordered sets (totally ordered sets, where every subset has a least element). This means (essentially) the ordinals are defined by the ordinals which precede them.

We start the ordinals with $0, 1, 2, \ldots$. Then comes the least infinite ordinal $\omega = \mathbb{N}$. Then $\omega + 1 = \{ 0, 1, 2, \ldots, \omega \}$. Etc.

Definition 5.5. Let x be a surreal number. Then x is a Conway ordinal number if it has an expression of the form $x = \{ L \mid \, \}$.

There is a technical result need that for any x, the CLASS of all ordinal numbers $\ngeq x$ is actually a set.

We then have the following theorem identifying Conway ordinal numbers with the usual ordinal numbers.

Theorem 5.6. *i) For each Conway ordinal* α, *we have* $\alpha = \{ \text{ordinals} < \alpha \mid \, \}$.
ii) In any non-empty CLASS of Conway ordinals, there is a least element.
iii) For any set S *of Conway ordinals, there is an Conway ordinal greater than every member of* S.

Be careful though that the operations on ordinals which match our surreal addition and multiplication are NOT addition and multiplication of ordinals. (E.g. ordinal addition is non-commutative since $1+\omega = \omega$.) The corresponding ordinal operations are so-called *natural sum* and *natural product*.

These operations come from adding/multiplying the Cantor Normal Form of an ordinal as a polynomial in ω. Recall

Theorem 5.7 (Cantor Normal Form)**.** *Every ordinal α can be written uniquely as*

$$\omega^{\beta_1} c_1 + \ldots + \omega^{\beta_k} c_k \, ,$$

where k is a natural number, $c_1, \cdots, c_k$ are positive integers, and $\beta_1 > \beta_2 > \cdots > b_k \geq 0$.

We will generalise this to obtain the Conway Normal form of a surreal number next.

CHAPTER 6

The structure of surreal numbers: birthdays, sign expansions, normal forms

1. Birthdays

We have (rather vaguely) talked about numbers constructed on Day 0, Day 1, etc. We can formulate this more precisely now by giving the birthday associated to any surreal number. This birthday turns out to be an ordinal number.

For an ordinal α, define M_α by $x = \left\{ x^L \mid x^R \right\} \in M_\alpha$ if all x^L, x^R are in $\bigcup_{\beta<\alpha} M_\beta$. Then

- M_α are the numbers born on or before day α (Made),
- $O_\alpha = \bigcup_{\beta<\alpha} M_\beta$ are the numbers born before day α (Old),
- $N_\alpha = M_\alpha \setminus O_\alpha$ are the numbers born on day α (New).

Theorem 6.1. *Every surreal number x is in a unique set N_α. We call α the birthday of x.*

PROOF. Suppose this is true for all x^L, x^R. Then pick some ordinal $\beta >$ the birthdays of all x^L, x^R. (Such an ordinal exists!). Then $x \in M_\beta$, so is created on some day $\alpha \le \beta$, i.e. is in some N_α. □

For $x \in N_\alpha$, we can consider approximations to x formed by taking only elements of O_β, $\beta < \alpha$. Lecture 10 21/12/2016

$$x_\beta = \{ y \in O_\beta, y < x \mid y \in O_\beta, y > x \}$$

Then $x_\alpha = x$, using the simplicity theorem. (Check this) And $x_\gamma = x$ for all $\gamma \ge \alpha$.

In some sense, x_β is the best approximation to x one can get on day β.

[Say $x_\beta < x$. If $z \in M_\beta$, $z = \left\{ z^L \mid z^R \right\}$, and $z < x$, then each $z^L < x$, and is in O_β. So already each z^L appears in the left set of x_β, meaning $z^L < x_\beta$. Moreover, $z < x < (x_\beta)^R$. So indeed we get $z \le x_\beta$. Therefore it suffices to only include the x_β when constructing x

$$x = \{ x_\beta < x \mid x_\beta > x \} ,$$

since any other option created on day β provides a weaker bound.]

For example, the approximations to $x = 5/8$ are

$$x_0 = \{ \mid \} = 0$$
$$x_1 = \{ 0 \mid \} = 1$$
$$x_2 = \{ -1, 0 \mid 1 \} = 1/2$$
$$x_3 = \{ -2, -1, -1/2, 0, 1/2 \mid 1, 2 \} = 3/4$$
$$x_4 = \{ \dots, 1/2 \mid 3/4, \dots \} = 5/8 = x$$

For example the approximations to $e = \exp(1)$ are

$$0, 1, 2, 3, 2\frac{1}{2}, 2\frac{3}{4}, 2\frac{5}{7}, \ldots .$$

The approximations for $\omega - 1$ are

$$0, 1, 2, \ldots, \omega, \omega - 1 .$$

The approximations for $\omega/2$ are

$$0, 1, 2, 3, \ldots, \omega, \omega - 1, \omega - 2, \ldots, \omega/2 .$$

The approximations for $\omega + 1/2$ are

$$0, 1, 2, \ldots, \omega, \omega - 1, \omega + 1/2 .$$

This previous theorem shows that the approximations x_β 'converge' to x, in a precise way. (They all coincide for some sufficiently large β.) We use this to define the sign expansion.

2. Sign expansion

Let x be a surreal number with birthday α, with approximations x_β. We define $s_\beta =$ sign of $x - x_\beta$, with $s_\beta = 0$, for $\beta \geq \alpha$. Notice that if x has birthday α, then $x - x_\alpha = 0$, so the α-th sign is 0. Comparing lengths of sign expansions leads to the previous notion of created earlier/simpler, x is created earlier than y if x has a shorter sign expansion than y.

We can lexicographically order these sign sequences, using $- < 0 < +$. I.e.

$$(s) < (t) \text{ iff the first different place has } s_\alpha < t_\alpha .$$

This associates a sequence of signs $+, -$ to x, which is $+, -$ below some ordinal α, and 0 (at or) beyond. We'll see that this turns out to be a bijective order preserving correspondence.

$$\begin{aligned} 5/8 &= + - + - \\ e &= + + + - + + - \cdots \text{(of length } \omega) \\ \omega - 1 &= +^\omega - \\ \omega/2 &= +^\omega -^\omega \\ \omega + 1/2 &= +^\omega - + . \end{aligned}$$

[Notice, we count from position 0, so $+^4 = ++++$ vanishes at position 4. Similarly, $+^\omega$ is $+$ for all positions $< \omega$, and is 0 at position ω. We can just sum the exponents of $+, -$ to get the birthday.]

These sign sequences justify the tree of surreal numbers we previously drew, and shows how it extends to all ordinal depths. $+$ means move to the right branch, and $-$ means move to the left branch from the current node.

The sign expansion for x_β is obtained by truncating the sign expansion for x to length β, and filling with 0's for positions $\geq \beta$. Check this (!)

Theorem 6.2. *Let (s), (t) be sign sequences for x, y. Then $(s) < (t)$, etc if and only if $x < y$, etc. So surreal numbers and sign sequences have the same ordering.*

PROOF. Suppose that $(s) < (t)$ with the first difference at α. That is to say, $s_\beta = t_\beta$ for all $\beta < \alpha$, but $s_\alpha < t_\alpha$.

Since x_β is obtained by truncating s, and setting $s_\gamma = 0$, for $\gamma \geq \beta$, we can say that $x_\beta = y_\beta$, for $\beta \leq \alpha$ by induction. (The position α is determined by the approximation x_α, it is not used in it. So the truncations $s(\alpha) = t(\alpha)$.)

Now $s_\alpha = \operatorname{sgn}(x - x_\alpha) < t_\alpha = \operatorname{sgn}(y - y_\alpha)$. Therefore

$$x - x_\alpha < y - y_\alpha \implies x < y\,,$$

since $x_\alpha = y_\alpha$. Alternatively, we can see this as $x < x_\alpha = y_\alpha < y$ by the signs s_α and t_α respectively.

[Viz:

$$x = 2\frac{1}{2} \leftrightarrow +++- = s$$
$$y = 4 \leftrightarrow ++++ = y$$

With approximations $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 2\frac{1}{2} = x$, and $y_0 = 0, y_1 = 1, y_2 = 2, y_3 = 3, y_4 = 4$.

The first place they differ is place 3. Truncate to $s(3) = t(3) = {+}{+}{+} \leftrightarrow 3 = x_3 = y_3$. But $s_3 = - < t_3 = +$. And we do have

$$s_3 = \operatorname{sgn}(x - x_3) = \operatorname{sgn}(2\frac{1}{2} - 3) < t_3 = \operatorname{sgn}(y - y_3) = \operatorname{sgn}(4 - 3)\,,$$

giving

$$2\frac{1}{2} - 3 < 4 - 3 \implies 2\frac{1}{2} < 4$$

]

Moreover, given x, y with $(s) = (t)$ of common total length α, we have that

$$x = \{\, x_\beta < x \mid x_\beta > x \,\}$$
$$y = \{\, y_\beta < y \mid y_\beta > y \,\}\,.$$

But by induction every $x_\beta = y_\beta$, $\beta < \alpha$, since the truncated sign sequences are equal.

Each parent x_β agrees with some parent y_β. Moreover x_β is in the left set of x iff $x - x_\beta > 0$ iff $s_\beta = t_\beta = +$ iff $y - y_\beta > 0$, which is if and only if y_β is in the left set of y. That is to say, these two numbers are the same. □

Also every sign expansion which we can write down does occur

Theorem 6.3. *For an arbitrary sequence (s) of signs $+, -$, 0 after some ordinal α. Then there is a number with sign expansion x.*

PROOF. Write $s(\beta)$ to mean the sequence truncated at β, filed with 0's at positions $\geq \beta$.

By induction we can assume that for $\beta < \alpha$, such a number x_β with sign expansion $s(\beta)$ exists. Now consider

$$x = \{\, x_\beta, s(\beta) < s \mid x_\beta, s(\beta) > s \,\} \; .$$

Since each parent of x is born on day $\leq \beta$, we see that x is born, at most, on day $\beta + 1 \leq \alpha$.

These x_β are the approximations to x. Now the set up of x means that the sign $x - x_\beta$ is s_β exactly, for $\beta < \alpha$. Since $s(\beta)$ is a truncation of s, this they agree for all places in $< \beta$. At β we have $s(\beta)_\beta = 0$, so $s_\beta < 0$ iff $s < s(\beta)$ iff $x_\beta > x$ iff $x - x_\beta < 0$. I.e. $s(x)_\beta = s_\beta$, so that x has the right signs. □

All of the theory of surreal numbers can be developed from this point of view first. Harry Gonshor (An introduction to the Theory of Surreal numbers) develops surreal numbers using the sign expansion as the starting point.

Exercise 6.4. Investigate the sign expansion of integers, dyadic fractions, and more generally real numbers. Can you manage to read off the number from the signs, or the signs from the number?

Exercise 6.5. What is the sign expansion of $\frac{3}{4}\omega$. How does this relate to the expansion for $\frac{3}{4}$?

Exercise 6.6. Show that $\sqrt{\omega} = \left\{\, 1, 2, 3, \ldots, n, \ldots \,\middle|\, \omega, \frac{1}{2}\omega, \frac{1}{4}\omega, \ldots, \frac{1}{2^m}\omega, \ldots \,\right\}$. (Square it!) Hence work out the sign expansion for $\sqrt{\omega}$.

3. Conway normal form

3.1. Omega map. Firstly we need to define the function ω^x which somehow represents the 'ordinal power' of omega. (This is different to exponentiation of surreal numbers, via the analytic surreal exponential function exp, say.)

We call two positive surreal numbers *commensurate* if there exists a positive integer n such that $x < ny$, and $y < nx$. For example, 5 and 7 are commensurate, with $n = 2$. But 7 and $\omega - 1$ are not commensurate since $7n < \omega - 1$, for any n. And ω, and $\omega^{1/2}$ are not comensurate, since $n\omega^{1/2} < \omega$ for any n, since $\omega^{1/2} = \{\, n \mid \omega/n \,\}$. This is a (convex) equivalence relation, i.e. $x < z < y$ and x, y commensurate implies xz and yz are commensurate.

Now we want to take the simplest members of each equivalence class (called a leader), to define ω^x. I.e. $\omega^0 = 1$, the simplest lead of all. Then $\omega^{\pm}$ are the next simplest leader to the left and right of $\omega^0 = 1$, i.e. $\omega^1 = \omega, \omega^{-1} = \epsilon = 1/\omega$. So $\omega^{3/4}$ is the simplest leader between $\omega^{1/2}$ and ω^1. The formal definition is

Definition 6.7 (ω map).

$$\omega^x := \left\{\, 0, r\omega^{x^L} \,\middle|\, r\omega^{x^R} \,\right\} ,$$

where r ranges over all (Conway) real numbers.

Theorem 6.8. *Every positive number x is commensurate with some ω^y.*

Proof. Write $x = \left\{ 0, x^L \mid x^R \right\}$, with $x^{L/R}$ positive. By induction each $x^{L/R}$ is commensurate with some $\omega^{y^{L/R}}$. If x is commensurate with one of its parents, then we are done. So assume not. Well this means that x is not commensurate with any $\omega^{y^{L/R}}$. Since $\frac{1}{n}\omega^{y^L} < x^L < x$, some n, we get $\omega^{y^L} < nx$. Therefore we cannot have $x < n\omega^{y^L}$, so $n\omega^{y^L} \leq x$. Let $r \geq n$ real, then $\omega^{y^L} < nx \leq rx$. So $r\omega^{y^L} \leq x$ by no-commensurateness. So $r\omega^{y^L} < x$ for all real r. Similarly $r\omega^{y^R} > x$, for all real r. So

$$x = \left\{ x_L, r\omega^{y^L} \,\middle|\, x^R, r\omega^{y^R} \right\}$$

Since $x^L < n\omega^{y_L}$, some n, we can drop it. Similarly since $n\omega^{Y^R} < X^R$ we can drop it.

So

$$x = \omega^y\,,$$

for $y = \{ y_L \mid y_R \}$. □

Theorem 6.9. *The map ω^x satisfies*

- $\omega^0 = 1$
- $\omega^{-x} = \frac{1}{\omega^x}$
- $\omega^{x+y} = \omega^x\omega^y$

Proof. i) is just a trivial calculation. ii) follow from iii). So prove iii)

With $\omega^x = \left\{ 0, r\omega^{x^L} \,\middle|\, r\omega^{x^R} \right\}$, $\omega^y = \left\{ 0, s\omega^{y^L} \,\middle|\, s\omega^{y^R} \right\}$, we compute that

$$\begin{aligned} \omega^x\omega^y = \{0, r\omega^{x^L+y}, s\omega^{x+y^L}, r\omega^{x^L+y} + s\omega^{y^L+x} - rs\omega^{x^L+y^L}, \\ r\omega^{y^R+x} + s\omega^{y^R+x} - rs\omega^{x^R+y^R} | s\omega^{y^R+x}, r\omega^{x^L+y} + s\omega^{y^R+x} - rs\omega^{x^L+y^R}, \\ r\omega^{x^R+y}, r\omega^{x^R+y} + s\omega^{y^L+x} - rs\omega^{x^R+y^L}\} \end{aligned}$$

using the induction assumption.

Since $x^L + y, x + y^L > x^L + y^L$, we see that the maximum index is either $x^L + y$ or $x + y^L$. Therefore $r\omega^{x^L+y} + s\omega^{y^L+x} - rs\omega^{x^L+y^L}$ is $\leq (r+s)\max(\omega^{x^L+y}, \omega^{x+y^L})$, so we can drop it. Since $\omega^x \leq r\omega^{x_R}$ any positive real r (so if $x < y$, $\omega^x < r\omega^y$, any positive real r) we see that the RR-left parent is negative, so we can drop it.

On the right hand side, similarly, we can drop both LR and RL parents, giving

$$\omega^x\omega^y = \left\{ 0, r\omega^{x^L+y}, s\omega^{x+y^L} \,\middle|\, r\omega^{x^R+y}, s\omega^{x+y^R} \right\} = \omega^{x+y}$$

So the proof is complete. □

3.2. Conway normal form. Now we can describe the Conway normal form of a surreal number. Let x be a positive surreal number (use $-x$, if $x < 0$). Then we can pick ω^{y_0} commensurate with x

Since x is commensurate with ω^{y_0}, $\omega^{y_0} < nx$ and $x < n\omega^{y_0}$, for some positive integer n. So considering $L = \{ r \mid r\omega^{y_0} \leq x \}$, $R = \{ r \mid r\omega^{y_0} > x \}$, we see $n \in R$, and $-n \in L$. So one of L, R contains an extremal point r_0. (If $\sup L \notin L$, the $R \ni \sup L = \inf R$.)

So write

$$x = \omega^{y_0} r_0 + x_1 \,.$$

Then $-x < nx_1 < x$, for any integer n, so x_1 is small compared to x. (Why? Well if $nx_1 > x$ (wlog), then

$$x_1 > \frac{1}{n} x \,,$$

so

$$\omega^{y_0} r_0 = x - x_1 < (1 - 1/n) x \,,$$

so

$$\omega^{y_0} \frac{r_0}{1 - 1/n} < x \,.$$

Yet $r_0/(1 - 1/n) > r_0$, so r_0 was not extremal.)

We can repeat this with x_1, et cetera to get

$$x = \omega^{y_0} r_0 + \omega^{y_1} r_1 + \ldots + \omega^{y_n} r_n + x_{n+1} \,.$$

If any $x_i = 0$, we get a finite sum. Otherwise the expansion may continue for more than ω steps. What does this mean?

Suppose that for $\beta <$ some α, we have defined the β-term $\omega^{y_\beta} r_\beta$ of x. THen

$$\sum_{\beta < \alpha} \omega^{y_\beta} r_\beta$$

is define to be the simplest number whose β term is $\omega^{y_\beta} r_\beta$.

Now we can write

$$x = \sum_{\beta < \alpha} \omega^{y_\beta} r_\beta + x_\alpha \,,$$

some x_α.

If $x_\alpha = 0$, set the α-term to be 0. Otherwise the α-term is $\omega^{y_\alpha} r_\alpha$, where ω^{y_α} is commensurate with $\pm x_\alpha$, and $x_\alpha - \omega^{y_\alpha} r_\alpha$ is small compared to x_α. This defines the $|alpha$-term for all ordinals α.

Since $\sum_{\beta<\alpha} \omega^{y_\beta} r_\beta$ is the simplest number having the same β-term as x, for all $\beta < \alpha$, this partial sum belongs to M_γ, where γ is the birthday of x. The partial sums cannot all be distinct for all ordinals α, so eventually the α-term is 0 for some α, and all terms beyond. x is the simplest number having the same β term as x, for all $\beta < \gamma$!

Theorem 6.10 (Conway Normal Form)**.** *Every surreal number x can be written uniquely in the (Conway normal) form*

$$\sum_{\beta < \alpha} \omega^{y_\beta} r_\beta \,,$$

where α is some ordinal, the numbers r_β, $\beta < \alpha$ are non-zero reals, and y_β form a decreasing sequence of surreal numbers.

Every such form satisfying these conditions occurs, and the normal forms for distinct x are also distinct.

Basically surreal numbers can be written as ordinal length sums of powers of ω, with real coefficients. One can show that adding/multiplying surreal numbers is equivalent to adding/multiplying these normal forms in the obvious way as polynomials in ω.

In fact this leads to the result that surreal numbers $\cong \mathbb{R}((\omega^{\text{surreal numbers}}))$; the FIELD structure of surreal numbers can be obtained from the additive GROUP structure via the "Hahn series"/"Malcev-Neumann transfinite power-series construction" with monomials ω^x, as x varies over the surreal numbers.

We can group the terms of the series $\sum_{\beta<\alpha} \omega^{y_\beta} r_\beta$, according to $y_\beta < 0, = 0, > 0$ to see that every surreal number can be written as

$$x = \text{infinitesimal part} + \text{real part} + \text{infinite part}$$

CHAPTER 7

Combinatorial Games

Lecture 11 Cancelled

Lecture 12 18/01/2017

In this chapter we want to switch focus a bit, and explore combinatorial games using surreal numbers and ideas we have developed so far. We will focus on two specific games: Hackenbush and Nim.

Nim is an impartial game meaning the two players have exactly the same possible moves. This means that so games are *not* surreal numbers because they have the form $G = \{ S \mid S \}$ for some set S. And obviously the condition no $g_l \geq$ any g_r fails since ever g_l is equal to some g_r. These games have a very deep and interesting theory behind them.

However Hackenbush is a partisan game, meaning the two players have different possible moves. In the restrained version of Hackenbush every game we encounter is actually a *number*! Whereas the unrestrained version is an impartial game, and equivalent to some nim game. A more general version of Hackenbush (again partisan) subsumes these both.

We will start with the restrained version of Hackenbush because it has (in some sense) the simplest theory.

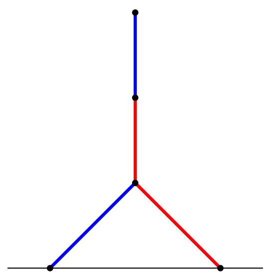
1. Hackenbush restrained

Definition 7.1 (Hackenbush restrained)**.** Hackenbush is a game played on a picture (graph) with some blue edges and red edges joining nodes. Each node is connected by a path to the *ground* (a certain dotted line).

On his turn, left (bLue) may chop a blue edge, whereas right (Red) may chop a red edge. After an edge is chopped, it disappears as does any edges which are no longer connected to the ground.

The game ends when no edge remains to be chopped, and the player who is unable to move is the loser.

Let's play a simple game of Hackenbush.



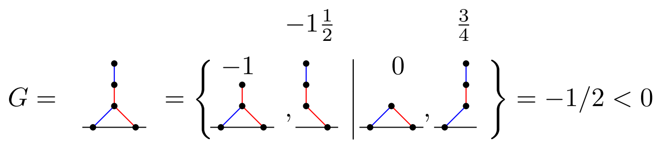
I'll play red, and you play blue. Do you want to go first, or second?

It looks like the right player can always win! Recall we introduced the following relations

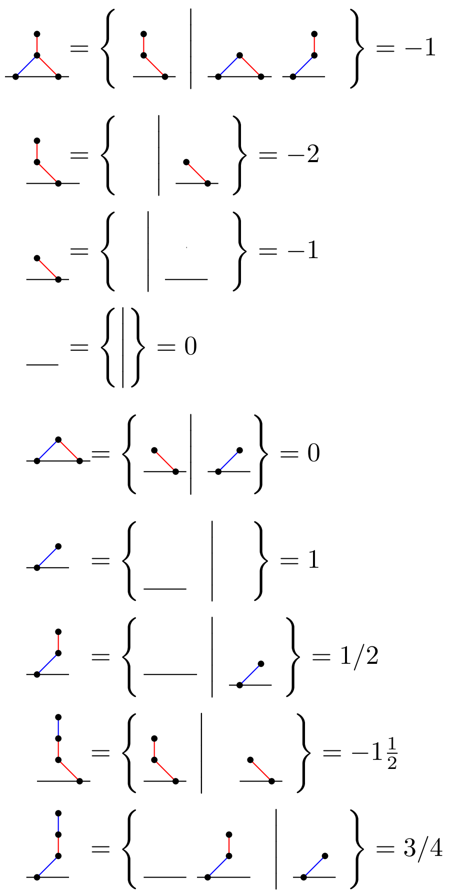
- $G > 0$ if there is a winning strategy for left
- $G < 0$ if there is a winning strategy for right
- $G = 0$ if there is a winning strategy for player 2
- $G \parallel 0$ if there is a winning strategy for player 1

So this game of Hackenbush is < 0. Does this mean it has a numerical value? If so, what is it? (Not necessarily, since $\{ -1 \mid -3 \}$ is ≤ 0. Check the definition of $\leq$, or use the above interpretation! Yet $\{ -1 \mid -3 \}$ is not even a number!)

Let's play through the game exhaustively.



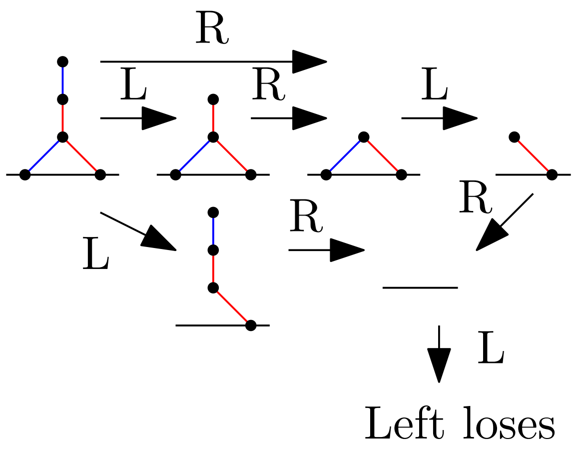
So now, let's consider the subsequence positions



We find that $G = -1/2 < 0$, so yes there should be a winning strategy for Right. What is this strategy? It's built up recursively using that fact that all Hackenbush positions are numbers! We see above that G is equal to a number, as is each subsequence position. This is a general result for Hackenbush.

We can give the following winning strategy: Since all left options are < 0, left can only move to a position where right wins. Some right options are ≤ 0. If there is a right option < 0, and it is right's move, move to it, and right still wins. Otherwise move to option 0, then it is left's move. But we are in a position where the second player (i.e. right) wins!

(One can also see how to potentially justify the relations $<, >, =, \|$ recursively by giving such strategies.)



I claimed above that every Hackenbush position is a number. How do we see this? Let's see first how to interpret $+, -$ on games. We have the formulae from previous, but we can see

$$-G = \{ -G_R \mid -G_L \} ,$$

as playing the game G but with the left and right players swapped. If I play left in G and you play right, then in $-G$ I play right, and you play left. (In Hackenbush, just swap the color of the edges)

Also in

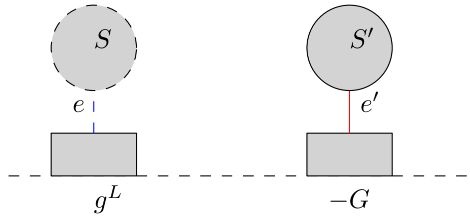
$$G + H = \left\{ g^L + h, g + h^L \mid g^R + h, g + h^R \right\} ,$$

we see the game $G + H$ is played by picking one and only one of G or H to play in each turn. In Hackenbush, we just draw the games G, H next to each other in the game playing field.

Theorem 7.2. *i) On chopping a blue edge, the value decreases, on chopping a white edge, the value strictly increases.*
ii) The value of every Hackenbush position is a number

PROOF. Let $G = \{ G_L \mid G_R \}$ be a Hackenbush position. We want to show that $g^L < G$, and $G < g^R$. It will then follow that G is a number.

Suppose g^L is obtained by removing some blue edge e and any now disconnected edges S. To show $g^L < G$, we show that $g^L - G < 0$, i.e. in the game $g^L - G$, there is a winning strategy for right.

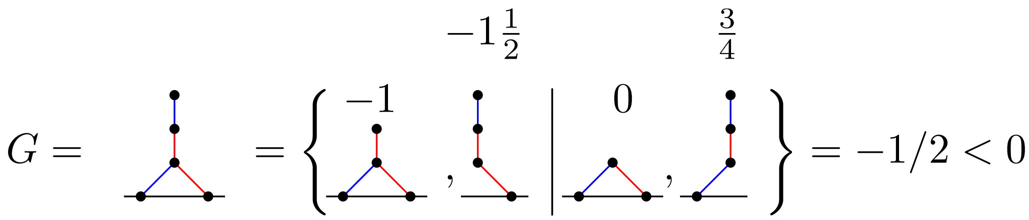


If right moves first, he should delete the corresponding (red) edge e' in $-G$ moving to the position $g^L - g^L = 0$. This is now a win for right since he plays second. He can win by always copying left's move in the other component.

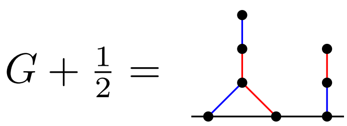
If left moves first, and plays in g^L right simply copies the move with the corresponding one in $-G$. For every move left makes, right has a response so right cannot lose here. Similarly if the move is in $(-G)$ but not in $S' \cup e'$. If the move is in $S' \cup e'$, then the move cannot be at e' since this is red, so the move is in S'. Now right simply cuts at e' reducing to $g^L - g^L = 0$ and wins as before.

In either case right wins, so $g^L - G < 0$, which shows $g^L < G$. By symmetry, we get $G < g^R$, and the theorem is proven. □

Knowing that



we perhaps might think that the following Hackenbush position is equal to 0, and so can be won by the second player. Try it and see!



Lecture 13
25/01/2017

1.1. Evaluating Hackenbush positions. We know every Hackenbush position has a numerical value. But how can we find it? For single stalks we see there is some strong similarity to the sign expansion: blue is +, red is −.

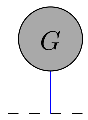
Fact 7.3 (Evaluating sign expansions of real numbers, Berlekamp)**.** The sign expansion $s_1 \cdots s_n$ can be evaluated as follows. We ca assume $s_1 = +$. If it consists of only +'s, it is n. Otherwise bracket the first $+-$. The +'s before this is the integer part. The signs after this form the binary expansion using $+ \leftrightarrow 1$ and $0 \leftrightarrow -$. If the length of s is finite, append a final 1.

E.g. $++++++-+--+- = ++++(+-)+--+- \to 4 + 0.10010[1]_2 = 4 + \frac{1}{2} + \frac{1}{16} + \frac{1}{64} = 4\frac{37}{64}$.

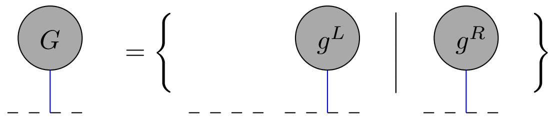
Also $(+-)^\omega = (+-)+-+-+-\cdots = 0 + .101010\cdots_2 = \sum_{i=0}^{\infty} \frac{1}{2}\frac{1}{4^i} = \frac{2}{3}$.

For trees there is a nice theory. But in general there finding the value of a red-blue Hackenbush position is NP-hard (See Chaper 7 of Winning Ways: Volume 1).

Let's consider how the value of



is related to the value of G. What positions can left move to, and what positions can right move to? We see



.

To describe this we introduce the following notion.

Definition 7.4 (Ordinal sum)**.** The ordinal sum $1 : x$ of 1 and x is defined by

$$1 : x \equiv \left\{ 0, 1 : x^L \;\middle|\; 1 : x^R \right\} .$$

Theorem 7.5. *If x is a real number, then $1 : x$ can be obtained by the first value from*

$$\frac{x+1}{1}, \frac{x+2}{2}, \frac{x+3}{4}, \ldots, \frac{x+n}{2^{n-1}}, \ldots$$

where $x + n > 1$.

So this does not depend on the form of x, only the value.

Proof. We only sketch the proof. From the expression above, we see that $1 : x$ is positive. And also $1 : x^L < 1 : x < 1 : x^R$. So $1 : x$ maps all surreal numbers to positive surreal numbers, in order of simplicity.

We see $0 \mapsto 1$, $1 \mapsto 2$, and generally integer $\ell \mapsto \ell + 1 = \frac{\ell+1}{2^0}$. Similarly $-1 \mapsto \frac{1}{2}$, $-2 \mapsto \frac{1}{4}$, and generally $-\ell \mapsto \frac{1}{2^\ell} = \frac{-\ell+(\ell+2)}{2^{\ell+1}}$.

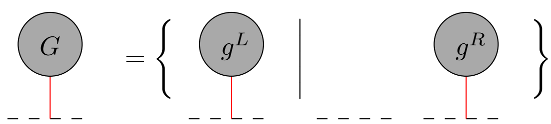
x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$1 : x$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4	5	6

Where do the other numbers fit in? Well the simplest number between -3 and -2 is halfway between them, and this maps to the simplest number between $\frac{1}{8}$ and $\frac{1}{4}$, also halfway between them. This holds generally, so that the intervals $n, n + 1$ are mapped linearly onto the image intervals.

Every $x \geq 0$ is just shifted by 1, $x \mapsto x + 1$. If $-1 \leq x \leq 0$, we map it to $[1/2, 1]$ by $x \mapsto \frac{x+2}{2}$. If $-2 \leq x \leq -1$, we map it to $[1/4, 1/2]$ by $x \mapsto \frac{x+3}{4}$, and so on. This gives the formulae above.

[We see that this breaks down for non-real numbers, as $1 : \omega = \{ 0, 1 : n \mid \ \} = \omega$.] $\square$

Similarly we can evaluate



using

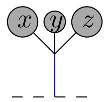
$$-1 : x \equiv \left\{ -1 : x^L \;\middle|\; 0, -1 : x^R \right\}$$

the ordinal sum of -1 and x. This can be otained by the first value from

$$\frac{x-1}{1}, \frac{x-2}{2}, \frac{x-3}{4}, \ldots, \frac{x-n}{2^n}, \ldots$$

where $x - n < -1$.

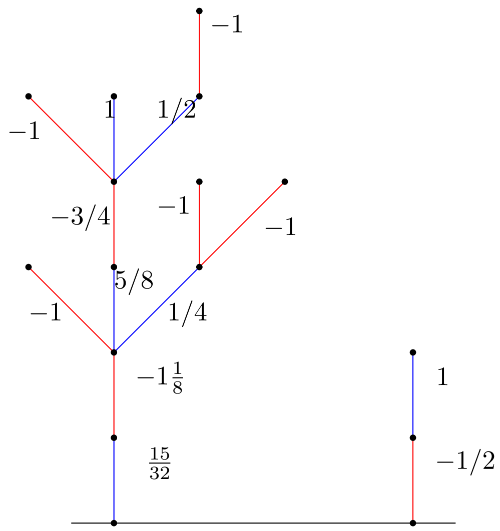
We have rules like the following



$= 1 : (x + y + z)$

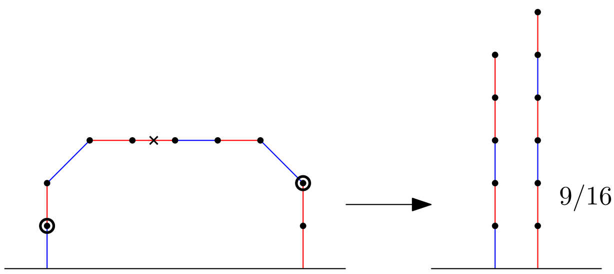
With this we can evaluate any sum of trees to determine the winner. The value written on each edge is the value of the tree obtained by removing any edges lower down then the current one.

$-1/8 =$



So in this position, the game G has value $-1/32 < 0$, so this is a win for right. (But only just!)

Berlekamp also gives a also rule for evaluating certain loops. Find the two sign changes nearest the ground. Cut the loop at exactly midway between these two points (either at a vertex, or in the middle of an edge which generates two new edges!). Sum the two resulting values.



$9/16 - 27/16 = -18/16$

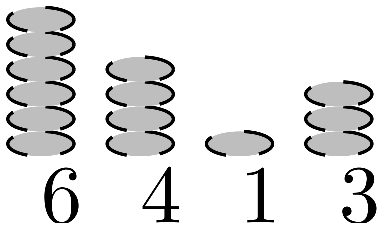
2. Nim

Lecture 14
01/02/2017

Definition 7.6 (Nim)**.** Nim is a game played with some number of heaps of matchsticks (or coins, etc). On each turn, a player may remove any positive number of matchsticks from exactly one pile so as to strictly decrease the number of stick in that pile. The sticks are discarded.

If a player has no moves remaining, then he loses.

Let's play a game of nim. I'll be polite and let you go first.



Let us try to analyse this game using Conway's surreal numbers and games.

If we have a pile of size n, let's write $*n$ for it. We can express the moves from position $*n$ as follows

$$*n = \{ *0, *1, \dots, *(n-1) \mid *0, *1, \dots, *(n-1) \} .$$

Since nim is an impartial game, the left and right options are always equal, so let's abbreviate this as

$$*n = \{ *0, \dots, *(n-1) \} .$$

For $n \in \mathbb{N}$, we call these things *nimbers*.

Playing with multiple piles is simply adding the corresponding piles as games, so the above game is

$$*6 + *4 + *1 + *3 ,$$

whatever that evaluates so.

Moreover, we see that $*0 = \{ \ \} = 0$, and $*1 = \{ *0 \} = \{ 0 \mid 0 \} = *$ (we previously called this game $*\dots$). Then $*2 = \{ 0, * \mid 0, * \}$

2.1. Properties of nimbers. The behaviour of addition of nimbers is very unusual, compared to the what we know about adding numbers. We can of course use the same definition of addition...

Lemma 7.7. *Suppose $*n$ is a nimber. Then $*n + *n = *0 = 0$.*

Proof. There is a winning strategy for the second player in the game $*n + *n$: copy the first player's move. Therefore $*n + *n = 0$. ☐

An important theorem in the theory of impartial games is the Sprague-Grundy Theorem which says that every (short, i.e. with finitely many positions) impartial game is equal to some nimber. We therefore know that $*6 + *4 + *1 + *3 = *n$, for some n. But which?

First let's try to indicate the proof the Sprague-Grundy Theorem.

Theorem 7.8 (Precursor)**.** *Let G be an impartial game played with a finite collection of numbers from 0, 1, 2, ... Each move affects exactly one number, allowing any decrease and possibly some increases. The rules of the game ensure that the same always terminates. Then the outcome is equivalent to the corresponding nim game.*

PROOF. Whichever player has a nim winning strategy can use it without using the extra moves. If the opponent uses any of the extra moves, simply undo it returning to the previous configuration. But we are closer to the end of the game as stipulated by the rules.

Moreover, by undoing the extra moves if necessary and copying the moves in the other game (nim if G, or G if nim), we see that the second player wins $G-$nim game, so it is $= 0$. □

Theorem 7.9 (Sprague-Grundy)**.** *Any (short) impartial game is equal to some nimber/nim-heap*

PROOF. We use induction. Suppose $G = \{ G_1, G_2, \dots \}$. Then the theorem is true for each G_1, so we have $G_i = n_i*$. Hence

$$G = \{ n_i* \} \ .$$

Let $n =$ minimal excluded element$n_1, n_2, \dots$ be the smallest number which does not appear in $n_1, n_2, \dots$. We claim that $G = n*$.

This game is played with the single number n. Any decrease is possible since every number $1, 2, \dots, n-1$ is not excluded (it is smaller than the minimal excluded one!). Some increases are possible, but we cannot move to n.

Using the previous theorem we see that G is equal to the nim pile $*n$. □

This $*n$ is called the (Sprague-)*Grundy* number of G.

From this we can work out how to add nimbers. In the example above we have

$$*1 + *3 = \{ *1 + *0, *1 + *1, *1 + *2, 3 + *0 \} = \{ *1, *0, *1 + *2, *3 \}$$

Then

$$*1 + *2 = \{ *1 + *1, *1 + *0, *2 + *0 \} = \{ *0, *1, *2 \} = *3\,,$$

meaning

$$*1 + *3 = \{ *1, *0, *3, *3 \} = *2$$

Then

$$\begin{aligned} *4 + *2 = \{ *4 + *0, *4 + *1, *2 + *0, *2 + *1, *2 + *2, *2 + *3 \} \\ = \{ *4, *4 + *1, *2, *3, *0, *2 + *3 \} \\ = \{ *4, *5, *2, *3, *0, *1 \} = *6\,. \end{aligned}$$

(Since $*4{+}*1 = \{ *4 + *0, *1 + *3, *1 + *2, *1 + *1, *1 + *0 \} = \{ *4, *2, *3, *0, *1 \} = *5$.)

So finally $*6 + *6 = *0$. And this is why I won the game we played at the start!

More generally

$$*a + *b = \text{mex}\left\{ *a' + *b, *a + *b' : a' < a, b' < b \right\},$$

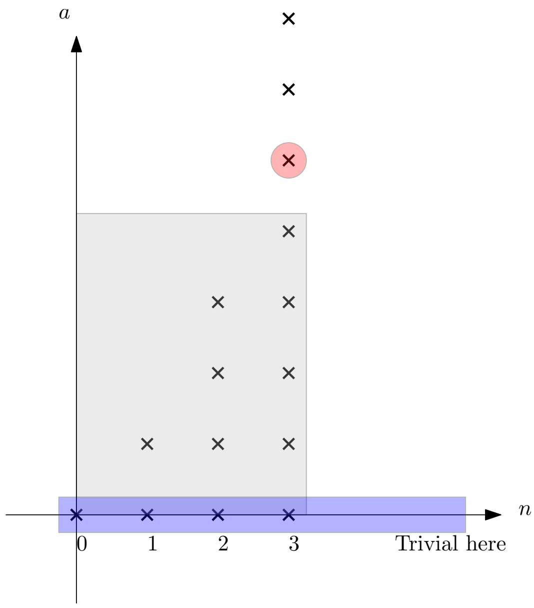
where mex means the minimal excluded element, minimal in the sense of the grundy number $*n \to n$.

2.2. XOR addition of nimbers. We can add nimbers like this, but it is time consuming. Is there a better way? The intuition from Sprague-Grundy is that nimber addition is the first thing it can possibly be, and this turns out to be xor addition – start writing down a possible addition table and see. Lecture 15 08/02/2017

Theorem 7.10 (Nimber addition). *Let $a, n \in \mathbb{Z}_{\geq 0}$.*

- *If $2^n > a$, then $*2^n + *a = *(2^n + a)$.*
- *Nim addition of $*a$ and $*b$ is binary addition without carry*
- *If $a, b < 2^n$, then $*a + *b = *c$, with $c < 2^n$.*

Proof sketch. To show the first point, we show that the second player wins in the game $*(2^n + a) + *2^n + *a$, hence it is $= *0$. This is by induction on n and on a.



For $n = 0$ or $a = 0$ the result is trivial. For $n = 0$, we must have $a < 2^0 = 1$, so $a = 0$ anyway. And for $a = 0$, we certainly have

$$*2^0 = *0 = *(2^0 + 0).$$

So now, assume we know that $*2^k + *b = *(2^k + b)$ for all $k < n$ and $b < 2^k$. And that $*2^n + *a' = *(2^n + a')$ for all $a' < a$. We want to use these, and the other properties proven simultaneously, to show that $*2^n + *a = *(2^n + a)$.

Suppose player 1 makes the following move, we give the reply which moves us back to position $*0$.

- $2^n + a \to 2^n + a'$, then move $a \to a'$.
- $2^n + a \to b$, $b < 2^n$, then move $2^n \to c$ where $*a + *b = *c$ since $c < 2^n$ by third point.

- $a \to a'$, then $2^n + a \to 2^n + a'$.
- $2^n \to b$, then move $2^n + a \to c$, where $*a + *b = *c$, since $c < 2^n$ by third point.

Then we see by induction that this is binary addition without carrying, because of how the leading digit behaves. This is also called the XOR of the binary numbers (for computer scientists/programmers).

Then from this we see $*a + *b = *c$, where $c < 2^n$ because we cannot possibly set the 2^n bit since neither a nor b has a 2^n bit. □

We can check out earlier result

$$*6 + *4 + *1 + *3 \to (110 \operatorname{xor} 100 \operatorname{xor} 001 \operatorname{xor} 011) = (000) \to 0*$$

2.3. Winning strategy. So we can determine the winner and the winning strategy easily now. Let $G = *n_1 + \cdots + *n_i$ be a nim-game.

If $G = *0$, then the second player wins. Otherwise $G = *n$, $n > 0$, and the first player wins. sThe winning strategy is to move back to the position $*0$. This is always possible:

Let $G = *N$ be the nim-sum of G. We compute

$$*n_i + *N = *m_i$$

At least one of the m_i is $< n_i$. Why? It is easiest to see this with the binary xor result. Find the most significant bit of N, say it is d. There must be a pile with most significant bit in position d, to make it so in N. Let this be pile n_1, wlog. Then $*n_1 + *N = *m_1$. We have $m_1 < 2^d$, but $n_1 \geq 2^d$. Hence $m_1 < n_1$.

Make this move to reduce to position $*0$. This is a position where the second player, and you are second to move.

Remark 7.11. Since either the first player wins, or the second player wins any nimber $*n$, we have either $*n = 0$, or $*0 \parallel 0$. In particular, nimbers are not positive, or negative! They are 0 or fuzzy.

2.4. Multiplication of nimbers. It is possible to define a multiplication on nimbers. Or rather the multiplication definition we have from before is well defined on nimbers.

This (amazingly) turns the nimbers into a FIELD of characteristic 2. In fact the nimbers $*0, \ldots, *(2^{(2^n)} - 1)$ form a subfield of order $2^{(2^n)}$. The overall result is an algebraically closed field of characteristic 2, which Conway calls On_2.

3. Miscellaneous games: Nim variants, RBG Hackenbush, ...

3.1. Nim variants, impartial games. We can use the Sprague-Grundy theorem to analyse some nim variants.

Example 7.12. Suppose we play nim, but we are only allowed to remove 1, 2, or 3 counters each turn. Who wins? We can recursively work out the Spragu-Grundey number of a size n pile $[n]$. We see

$$\begin{aligned}
[0] &= \{\ \} = *0 \\
[1] &= \{\ [0]\ \} = \{\ *0\ \} = *1 \\
[2] &= \{\ [0], [1]\ \} = \{\ *0, *1\ \} = *2 \\
[3] &= \{\ [0], [1], [2]\ \} = \{\ *0, *1, *2\ \} = *3 \\
[4] &= \{\ [1], [2], [3]\ \} = \{\ *1, *2, *3\ \} = *0
\end{aligned}$$

Generally this is periodic with period 4, so the Sprague-number is given by

$$[n] = *(n \mod 4)\,.$$

Who wins the game $[5] + [7] + [2]$, and what is the winning move? Well

$$[5] + [6] + [2] = *1 + *2 + *2 = *1\,,$$

so the first player wins. To find a winning move, we must find a move which moves to position *0. We find that

$$[5, 6, 2] \to [4, 6, 2]$$

is such a move. (Check that player 1 has a winning response to any move by player 2.)

3.2. Hackenbush unrestrained.

Definition 7.13 (Hackenbush Unrestrained)**.** Hackenbush unrestrained is played in the game way as Hackenbush restrained, but all edges are green. Green edges can be cut by either player.

Since both players have the same possible moves, this is an impartial game. By the Sprague-Grundy theorem, any Hackenbush unrestrained game is equivalent to some nimber $*n$.

Principles for playing Hackenbush unrestrained:

Fact 7.14.
i) Colon principle: when branches come together at a vertex, one can replace the branches by a stalk with the same nim sum.
ii) Fusion principle: vertices on any loop can be fused without changing the Sprague-Grundy number.

With these in mind, we can outline a procedure for finding the Sprague-Grundy number of any Hackenbush unrestrained position.

Fact 7.15. The Grundy number (also called the weight) of Hackenbush unrestrained game can be found as follows.

Identify all vertices in a loop to a single vertex, and collapse all edges to loops at a node. (This is only to compute the value!) The loops can be replaced with leaves (in the sense of tree graphs).

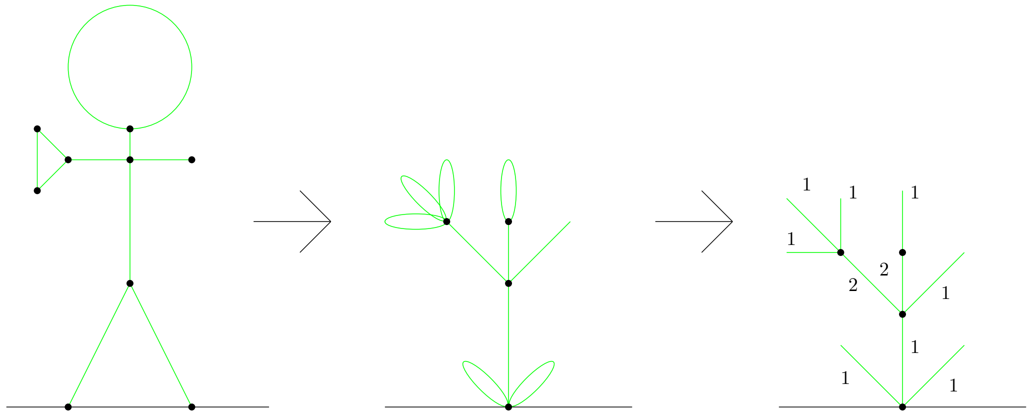
Starting from the leaves, write the number $(a_1 +_2 a_2 +_2 \cdots +_2 a_n) + 1$ on each edge, where a_i are the numbers on the edges immediate above the current edge and $+_2$ means the nim-sum. These numbers give the *stress* $\sigma(x)$ on edge x.

The weight of an edge is $(\sigma(x)|0)$ where

$$(x|y) = \begin{cases} 2^{n+1} - 1 & x \equiv y \text{ modulo } 2^n \text{ but not modulo a higher power of 2.} \\ -1 & x = y \end{cases}$$

The weight of the picture is given by the sum over weights of all root edges.

Example 7.16. we apply it to the following 'stick man'



The root stresses are $1, 1, 1$. But $(1|0) = 2^1 - 1 = 1$, so the weights are also 1 and this game should be equivalent to $*1 + *1 + *1 = *1$. This means a win for the first player. Indeed: first player takes the body, and we are reduced to just the two legs $*1 + *1 = *0$. Whichever one player 2 takes, the player 1 can take the remaining one.

3.3. RBG Hackenbush.

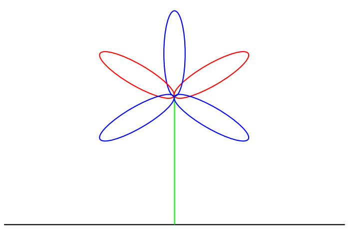
Definition 7.17 (RBG Hackenbush)**.** RBG Hackenbuhsh played in the game way as Hackenbush restrained, but some edges can be green. Green edges can be cut by either player.

On right's turn, he must cut either a red edge, or a green edge. On left's turn, he must cut either a blue edge, or a green edge.

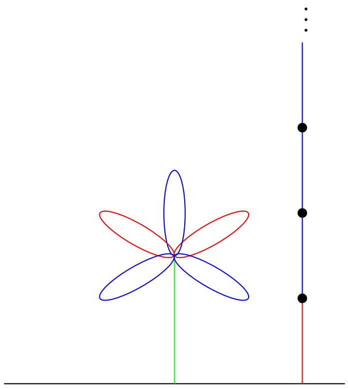
In some sense you can see RBG Hackenbush as a simultaneous generalisation of Hackenbush restrained, and of Nim. With no green edges, we have Hackenbush restrained. With only green edges (arranged into stalks), we have Nim or more generally Hackenbush unrestrained. So this is going to be more complicated than either of them!

One discovers some interesting positions in RBG Hackenbush

Example 7.18. In Winning Ways Conway gives the following as an example of a RBG hackenbush position. The flower G below is fuzzy: the first player wins by taking the green edge, so it is definitely $\neq 0$.



On the other hand, it is less than any positive number, and greater than any negative number. $G - \frac{1}{2^n} < 0$



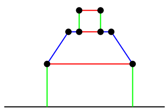
Right wins this as follows.

If right moves first: chop the green edge, then the position remaining is < 0, so right wins.

If left moves first, he either chops the green edge and right wins. Or he chops a petal, and right chops the green stalk. Otherwise he chops the Hackenbush stalk, and then right chops the green edge.

By using the sign expansion of a positive number to build the stalk, the result holds generally for any positive number. Flower $<$ any positive number.

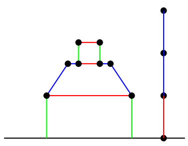
Example 7.19. More interesting is a positive Hackenbush RBG game, which is still less than all positive *numbers*. Conway gives the following house position G in Winning Ways.



$G > 0$ since left can win. The key point is which player cuts the first (green) wall. Whoever does this loses, since the other play cuts the remaining one and wins. Starting with the chimney, if left goes first he can can make at least 5 moves (one edge of the chimney plus 4 blue edges), whereas right can make only 4 moves (3 red plus the other chimney edge). If right goes first, then left can take the top two blue edges, leaving 2 blue to one 1 red.

So right must take the first green wall, and so lose.

However $G - \frac{1}{2^n} < 0$ because right can win.



If right goes first: take a green wall. If blue takes the other the result is < 0. If blue takes any roof edge the or a stalk edge then take the other green wall and the result is < 0.

Suppose left goes first. If he takes a green wall, take the other and the result is < 0. If he takes a roof or a stalk edge, take a green wall. (If blue takes the other it is < 0, if blue takes a root or a stalk, take the other and it is < 0.)

So this house G is > 0, but $<$ any positive surreal number! It is genuinely infinitesimal!

Like the house, the more abstract combinatorial game $\uparrow := \{\, 0 \mid * \,\}$ is a positive infinitesimal. This is but the start of a whole series of games which appear 'in the gaps' (Dedekind sections of surreal number), which Conway studies in On Numbers and Games and in Winning Ways.

CHAPTER 8

Potential further topics for games and for surreal numbers

This course only just scratches the surface surreal numbers, and combinatorial game theory. Some further reading can include:

i) Number theory with surreals: omnific integers and solutions to diophantine equations. Solutions to $x^2-(\omega+3)y^2=1$ give convergents to $\sqrt{\omega+3}$. E.g. $(2/3\omega+1)^2-(\omega+3)(2/3\sqrt{\omega})^2=1$. (ONAG, Gonshor)
ii) Integration, and analysis on surreals. Problems with defining integral.
iii) Games 'in the gaps' (Starts in ONAG, more in Winning Ways)
iv) Many different kinds of 'addition of games' (must play in all components, can play in any number of components, can play in only 1 component +) (ONAG Chp 14)
v) Deeper results for RBG Hackenbush: weakly order preserving functions: adding multiple wedges underneath.
vi) Temperature theory, thermographs, cooling games. (ONAG)
vii) Applications to 'real' games such as Go. (Mathematical Go)