NUMBERS! - PROBLEM SHEET 1

(1) Show that the two version of induction for $\mathbb{N}$ are equivalent. (Recall: Version 1: $M \subset \mathbb{N}$, with $0 \in M$ and for each $m \in M$, we have $S(m) \in M$ implies $M = \mathbb{N}$. Version 2: $\phi(n)$ a true/false statement about $n \in \mathbb{N}$. If $\phi(0)$ true, and $\phi(n)$ implies $\phi(n+1)$, then ϕ is true for all $n \in \mathbb{N}$.)

Use only the Peano axioms to solve the following problems. The order of the problems is important. You may find that some of the earlier problems are necessary to solve the later ones.

(2) Show that $+$ on $\mathbb{N}$ is commutative. (Hint: show first that 0 is an identity for $+$, so $0+a = a+0 = a$, for any $a \in N$. Then show the case $a+1 = 1+a$ separately.)

(3) Show that 0 is the unique natural number for which $n + 0 = n$, for all $n \in \mathbb{N}$. That is, if 0^* is another number for which $n + 0^* = n$, for all $n \in \mathbb{N}$, show that $0^* = 0$.

(4) Show that $+$ on $\mathbb{N}$ is cancellative. That is show that if $x + y = x + z$, then $y = z$.

(5) Show that $\cdot$ on $\mathbb{N}$ is commutative nad associative.

(6) Let $1 := S(0)$. Show that 1 is the multiplicative identity: $1 \cdot n = n \cdot 1 = n$, for all $n \in \mathbb{N}$. Show also that 1 is the unique such.

(7) Prove that $\cdot$ on $\mathbb{N}$ is right distributive: $(n + m) \cdot p = (n \cdot p) + (m \cdot p)$, for all n, m, p.

(8) Prove that $\cdot$ on $\mathbb{N}$ is commutative. (Hint: show first that $0 \cdot n = n \cdot 0$, for all $n \in \mathbb{N}$.)

(9) Prove that $\cdot$ on $\mathbb{N}$ is associative.

Using the definition of $\mathbb{Z} := \mathbb{N} \times \mathbb{N}/\sim$, and the definition addition and multiplication on $\mathbb{Z}$, check the following.

(10) Check that $\sim$ on $\mathbb{N} \times \mathbb{N}$, where $(a, b) \sim (c, d)$ iff $a + d = b + c$, is an equivalence relation. (Check: reflexivity $x \sim x$, symmetry $x \sim y$ implies $y \sim x$, and transitivity $x \sim y$ and $y \sim z$ implies $x \sim z$.)

(11) Check that the addition and multiplication on $\mathbb{Z}$ is well defined. If $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$ check that $[(a, b)] + [(c, d)] = [(a', b')] + [(c', d')]$, et cetera.

(12) Show that $+$ on $\mathbb{Z}$ is commutative, and associative. What is the identity for $+$? Show it is unique. What is the additive inverse of $[(a, b)]$? Show that this is unique.

(13) Show that $\times$ on $\mathbb{Z}$ is commutative, and associative. Show that $+$ distributes over $\times$.

(14) Show that $\mathbb{Z}$ is an integral domain. If $a \cdot b = 0$, then either $a = 0$, or $b = 0$. (Assume it known that in $\mathbb{N}$, if $n \neq 0$, and $m \neq 0$, then $n \cdot m \neq 0$. Prove this using the ordering.)

Using the definition of $\mathbb{Q} := \mathbb{Z} \times (\mathbb{Z} \setminus \{ 0 \}) / \sim$, and the definition addition and multiplication on $\mathbb{Z}$, check the following.

(15) Check that $\sim$ on $\mathbb{Z} \times (\mathbb{Z} \setminus \{ 0 \})$, where $(a, b) \sim (c, d)$ iff $ad = bc$, is an equivalence relation. (Check: reflexivity $x \sim x$, symmetry $x \sim y$ implies $y \sim x$, and transitivity $x \sim y$ and $y \sim z$ implies $x \sim z$.)

(16) Check that the addition and multiplication on $\mathbb{Q}$ is well defined. If $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$ check that $[(a, b)] + [(c, d)] = [(a', b')] + [(c', d')]$, et cetera.

(17) Show that $\times$ on $\mathbb{Q}$ is commutative, and associative. What is the identity for $\times$? Show it is unique. What is the multiplicative inverse of $[(a, b)]$, $b \neq 0$? Show it is unique.

Using the definition of $\mathbb{R}$, as Dedekind cuts of $\mathbb{Q}$, check the following.

(18) Convince yourself that the addition, and multiplication of cuts reflects/generalises the addition and multiplication of rational cuts.

(19) Check that addition on $\mathbb{R}$ is commutative, and associative.

(20) Check that multiplication of (positive) cuts is associative. (You would have to check another 3 cases to prove that multiplication is associative in general. Don't bother.)

(21) How should you define the cut y^{-1}, for positive y? Convince yourself this is the multiplicative inverse to y.

(22) Read about the completeness property of $\mathbb{R}$, and how it follows from this construction of Dedekind cuts. For example, Chapter 2, Section 5.2 of Numbers (Ebbinghaus, et al).

(23) Read about the other constructions of $\mathbb{R}$, by Cauchy sequences modulo null sequences, or by nested intervals. For example, Chapter 2 of Numbers (Ebbinghaus, et al). Compare and contrast with the Dedekind cut version.