NUMBERS! - PROBLEM SHEET 3

(1) Complete the proof for the $y \not\leq x$, and $b \equiv \{ X_L \mid y, X_R \}$ has $b = x$ case.

(2) Suppose that $x = \{ X_L \mid X_R \}$ is a surreal number, and that X_L and X_R are finite sets. Show that $x = \{ \max X_L \mid \min X_R \}$, so only one element is necessary in the left set, and in the right set.

(3) Justify all of the equalities and inequalities on day 2 using these two results.

(4) Check (using exactly the same definition as for numbers) that $0 \not\leq \{ 0 \mid 0 \}$, and $\{ 0 \mid 0 \} \not\leq 0$. So games 0 and $\{ 0 \mid 0 \}$ cannot be compared. $\{ 0 \mid 0 \}$ is said to be 'fuzzy' against 0.

Below, we will analyse the situation on day n in more detail.

(5) Suppose that on day n, we have the following list of numbers

$$x_1 < x_2 < \cdots < x_m .$$

(a) Using the question (2), produce a list of all possible numbers that appear on day $n+1$.

(b) Show that $\{ \ \mid x_1 \}$, $\{ x_m \mid \ \}$ and $\{ x_i \mid x_{i+1} \}$ for $i = 1, \ldots, m-1$ are new numbers which do not exist on day n.

(c) How are these new and old numbers ordered?

(d) (Simplicity Theorem) Show that the remaining numbers already exist on day n. For the number $y \equiv \{ x_i \mid x_j \}$, consider the earliest created number(s) $x_{i+1} \leq z \leq x_{j-1}$. Show that $y = z$.

(e) What happens for $\{ x_i \mid \ \}$, $i = 1, \ldots, m-1$, or $\{ \ \mid x_j \}$, $j = 2, \ldots, m$?

Now using the definition of addition, we can determine the values of all of these numbers

(f) Suppose x is the largest number on some day, show that $x+1 = \{ x \mid \ \}$.

(g) Now suppose that $a < b$ are surreal numbers such that there is no $a < w < b$ older than either of them (created earlier than a, or created earlier than b). Show that $\{ a \mid b \} + \{ a \mid b \} = a + b$. Hence deduce the values of numbers on all finite days.