NUMBERS! - PROBLEM SHEET 4

(1) Show that the collection of all surreal numbers is too large to form a set, that is show surreal numbers form a proper CLASS.

(2) Investigate this operation

$$x \oplus y = \left\{\, x^L \oplus y^L \,\middle|\, x^R \oplus y^R \,\right\} .$$

(Recall this that might be a first naïve guess for how to define $+$.)

(3) For all surreal numbers x, y, z, show that the following identities/equalities hold using 'one-line' proofs.

(a) $x0 \equiv 0$,

(b) $x1 \equiv x$,

(c) $xy \equiv yx$,

(d) $(-x)y \equiv x(-y) \equiv -(xy)$,

(e) $(x + y)z = xz + yz$,

(f) $(xy)z = x(yz)$

Can we replace $=$ with $\equiv$ in e), or f)?

Recall that we proved multiplication of surreal numbers is well-defined under $=$. The same is not true for games.

(4) Show how the remaining three cases of $(xy)^L < (xy)^R$ in i) of the proof that xy is a number reduce to combinations of $P(x_1, x_2 : y_1, y_2)$.

(5) Show how the remaining cases of $(x_1y)^L < x_2y < (x_1y)^R$ in ii) of the proof that xy is a number reduce to combinations of P.

(6) Find a game g, such that $\{\, 1 \mid \,\}g \neq \{\, 0, 1 \mid \,\}g$, even though $\{\, 1 \mid \,\} = \{\, 0, 1 \mid \,\} = 2$.

Recall how we defined $y = \frac{1}{x}$ in lectures.

(7) (Challenge) Try to come up with a definition of $y = \sqrt{x}$, which works for non-negative surreal numbers.

(8) Recall $\omega = \{\, 0, 1, 2, \dots \mid \,\}$ is an 'infinite number', and $\epsilon = \left\{\, 0 \,\middle|\, 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$ is an 'infinitesimal number' (less than any real number). What is the value of $\epsilon\omega$? (It should be a very familiar number.)