NUMBERS! - PROBLEM SHEET 5

(1) Let x be a rational number, whose denominator divides 2^n. Show that
$$x = \left\{ x - \frac{1}{2^n} \;\middle|\; x + \frac{1}{2^n} \right\}.$$

Recall the definition that x is a Conway real: $-n < x < n$ for some integer n, and
$$x = \left\{ x - 1, x - \frac{1}{2}, x - \frac{1}{3}, \ldots \;\middle|\; x + 1, x + \frac{1}{2}, x + \frac{1}{3}, \ldots \right\}.$$

(2) Show, using the previous exercise, that all dyadic fractions $a/2^k$ are Conway real numbers.

(3) Show that if x, y are Conway real numbers, then so are $-x, x + y$ and xy.

The following results allow us to identiy Conway real numbers, with the usual real numbers constructed (via Dedekind cuts, say)

(4) Show that each Conway real hs a unique expression of the form $\{ L \mid R \}$, where L, R are non-empty sets of rationals, L has no greatest element, R has no least element, there is at most one rational not in $L \cup R$, and L is downwards closed ($y < y' \in L \implies y \in L$) R is upwards closed.

(5) Show that every such choice for L, R as above gives a (unique) Conway real number.

Now convince yourself that this identifies the usual real numbers with Conway real numbers (operations match, etc).

Recall now that x is a *Conway ordinal* if it has an experssion of the form $x = \{ L \mid \}$. The following results identify Conway ordinals with the usual ordinals.

(6) Let α be a Conway ordinal. Show that $\alpha = \{ \text{ordinals} < \alpha \mid \}$. (Be careful to justify that this is a surreal number.)

(7) Show that any class of Conway ordinals contains a least element. (Hint: Consider $\{ L \mid \}$, where L is the set (justify!) of all β which are $<$ all $\alpha \in C$.)

(8) Let S be a set of Conway ordinals. Show (by constructing it) that there is a Conway ordinal greater than every element of S.

Recall how the β-th approximation x_β to a surreal number x is defined

$$x_\beta = \{\, y \in O_\beta, y < x \mid y \in O_\beta, y > x \,\}$$

(9) Suppose $x_\beta < x$. Show that any $y \in M_\beta$ which satisfies $y < x$ actually satisfies $y \le x_\beta$.

(10) Show that the approximations of approximations are compatible, in the following sense

$$(x_\beta)_\gamma = x_\gamma \,,$$

where $\gamma \le \beta$. Hence deduce that the sign expansion of x_β is obtained merely by truncating $s(x)$, by setting positions all $\ge \beta$ to 0.

(11) (Open ended) Investigate the sign expansion of integers, dyadic fractions, and more generally real numbers. Can you manage to read off the number from the signs, or the signs from the number?

(12) What is the sign expansion of $\frac{3}{4}\omega$? How does this relate to the expansion for $\frac{3}{4}$? Try this for other rational multiples of ω.

(13) Show that $\sqrt{\omega} = \left\{\, 1, 2, 3, \ldots, n, \ldots \;\middle|\; \omega, \frac{1}{2}\omega, \frac{1}{4}\omega, \ldots, \frac{1}{2^m}\omega, \ldots \right\}$. (Square it!) Hence work out the sign expansion for $\sqrt{\omega}$. (Doesn't have to be completely rigorously, but try to justify the age of $\sqrt{\omega}$, for instance do we need every $\frac{1}{2^m}\omega$ in the right hand set? And when is each created?)