NUMBERS! - PROBLEM SHEET 6

The handout `https://www.math.uni-tuebingen.de/user/charlton/teaching/numbers_1617/handout1_justifyrelations.pdf` contains part of a proof by induction of the interpretations of $<, =, >, \parallel$ on games.

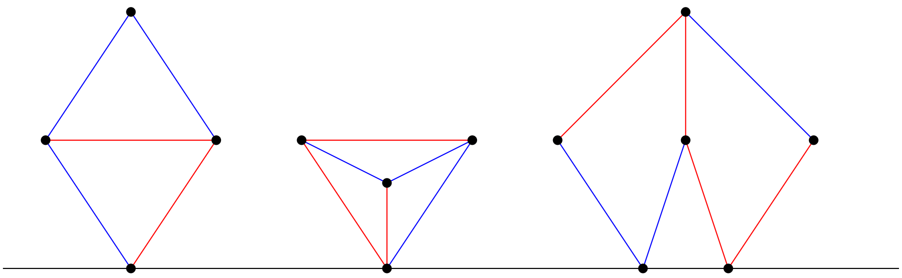
(1) Show the remaining cases $G > 0, G = 0$ and $G \parallel 0$ hold, and hence complete the proof.

Hackenbush

(2) (Open ended) Play some games of Hackenbush! Evaluate the positions to determine who wins, and then find the winning strategy.

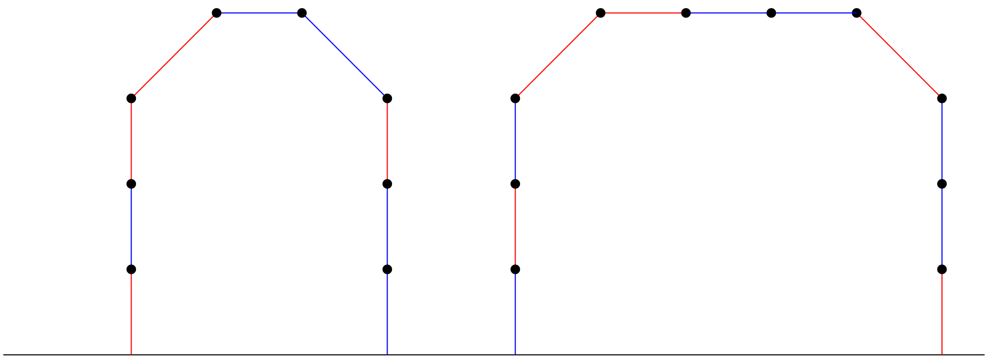
(a) Choose some 'simple' graphs (4 or 5 vertices, say), such as those below.

(b) Find a version of Hackenbush Restrained (also called Red-Blue Hackenbush) in your favourite app store.



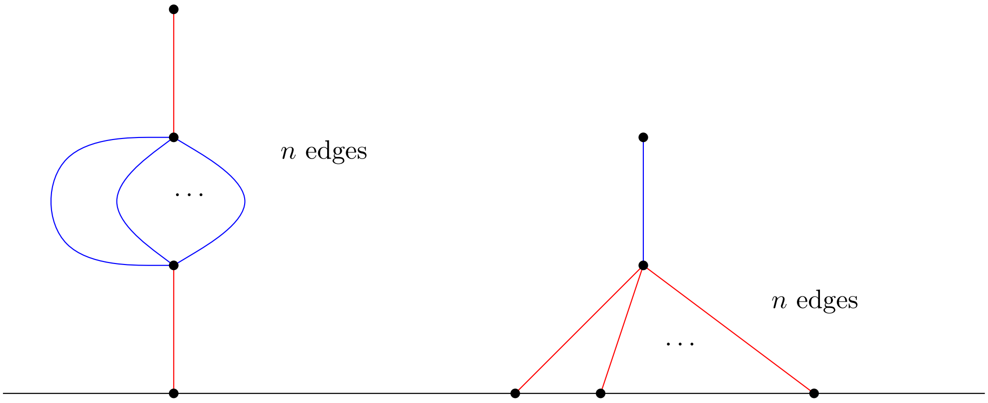
In the lectures, I gave a rule for evaluating loops/chains.

(3) Draw some loops such as those below and evaluate them directly, without using the rule. Check the rule gives the correct value.



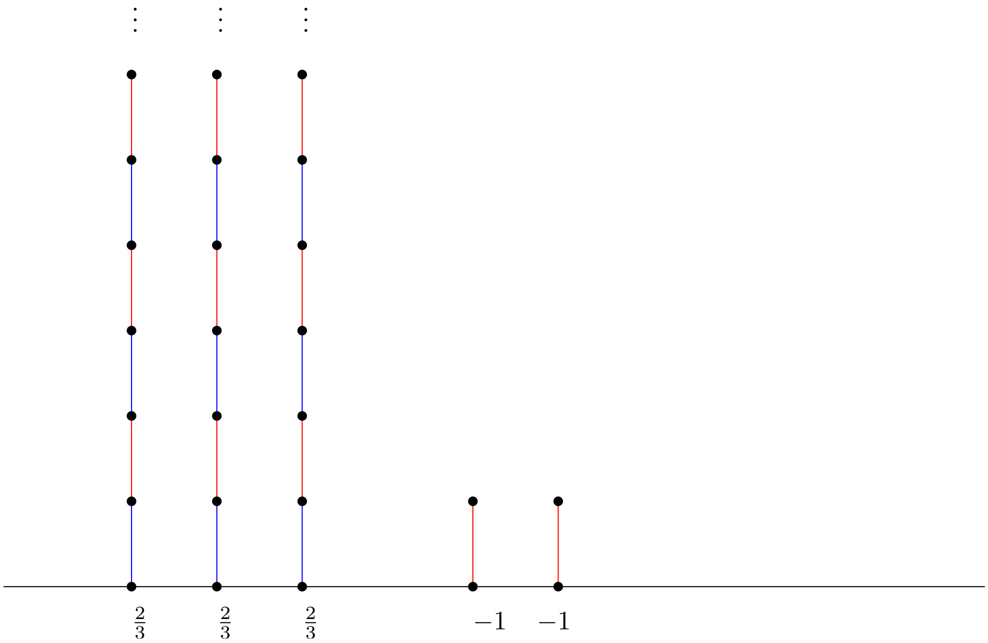
Beyond trees, and loop it is possible to evaluate some infinite families of Hackenbush positions.

(4) Find a formula for the value of each of the following positions, with proof.



It is possible to play Hackenbush on (certain) infinite graphs. An infinite stalk is fine: if the stalk has height ω, then the first move reduces it to a finite stalk. More generally, there is no infinite decreasing sequence of ordinals, so after a finite number of moves any stalk must vanish.

(5) The sign expansion rule claims that each of the ω-high stalks has value $\frac{2}{3}$. Justify this by finding a winning strategy for the second player.



Nim variants and impartial games

I mentioned that a consequence of the Sprague-grundy theorem is that nim addition is the ‘first’ thing it can possibly be. One can build up an addition and multiplication table for nimbers in such a way.

(1) Follow the strategy below to build up the nim addition and multiplication table for $*0, \ldots, *15$. Check that this agrees with the xor addition, and the the more explicit addition from lectures. Is there a nice 'rule' for multiplication? [See Chapter 6 of On Numbers and Games, for more details and solutions.]

Strategy: to fill in $*n + *m$ we must already have filled in $*n' + *m$ and $*n + *m'$, for $n' < n$, $m' < m$. The result for $*n + *m$ is the first thing which is consistent with the table being part of the addition table of a FIELD. To start: $*0 + *0 =?$. Well the smallest possible value is $*0$, and this is consistent since every field contains an element x with $x + x = x$, namely the zero element. Therefore also $*n + *0 = *0 + *n = *n$. So start with

$+$	$*0$	$*1$	$*2$	$*3$
$*0$	$*0$	$*1$	$*2$	$*3$
$*1$	$*1$	a	b	c
$*2$	$*2$	b	d	e
$*3$	$*3$	c	e	f

What about $a = *1 + *1$? It could be $*0$ since there are fields of characteristic 2, so it must be $*0$. Hence $*n + *n = 0$ also, so $a = d = g = *0$. Now $b = *1 + *2$ cannot be $*0, *1$ or $*2$ otherwise the table is no longer the addition table for a field. It could be $*3$, and so it must be.

We can use the Sprague-Grundy theorem to study any other impartial game, because any such game is equivalent to a game of nim.

(2) Suppose you play a nim variant where you may remove only a square number of counters $1, 4, 9, \ldots$ from some pile.
   (a) Work out the Sprague-Grundy number of a size n pile $[n]$ for $n = 0, 1, \ldots, 10$. Hint: the list begins $0, 1, 0, 1, 2, 0, 1, 0, 1, 2, 0$.
   (b) For the following games, determine the winning player. If player 1 wins, give the first move, or moves, of a winning strategy. If player 2 wins, bit the winning responses.
$$[3, 5, 7], [2, 4, 6, 7], [2, 2, 5], [3, 3, 4]\,.$$

(3) Repeat the above, but play a nim variant where you may remove only 1, 3 or 4 counters. Hint: the Sprague-Grundy list begins $0, 1, 0, 1, 2, 3, 2, 0, 1, 0, 1$.

(4) Repeat the above, but play a nim variant where you instead split a pile into some number > 1 of new piles, such that no two of the new piles have the same size. For example, you can move $10 \to [7, 2, 1]$, but not $\not\to [6, 2, 2]$. Hint: the Sprague-Grundy list begins $0, 0, 0, 1, 0, 2, 3, 4, 0, 5, 6$.

(5) Invent and analyse some impartial nim variants on your own.