Primes - Problem Sheet 1

Open ended questions for personal exploration

Try to get a feeling for some of the results discussed in the introduction lecture, by looking at the following questions. During the course, we will learn how to properly solve them. At the moment, no particular answers are expected!

Q1) 'Nice' congruence criteria exist for the following cases:

- $x^2 + 7y^2$ where discriminant $D = -28$,
- $x^2 - 5y^2$ where $D = 20$,
- $x^2 - 13y^2$ where $D = 52$,
- $x^2 + xy + 5y^2$ where $D = -19$,
- $x^2 + xy - y^2$ where $D = 5$,
- $x^2 + xy - 4y^2$ where $D = 13$.

Try to discover these conditions in one or two of the cases: which primes p can be written in these forms? What patterns do these p satisfy?

Perhaps you can write a computer program to investigate, or use a computer algebra system like Mathematica, Maple or Sage? Or even just a spreadsheet?

Q2) Recall the identity

$$\begin{aligned}&(2x^2 + 2xy + 3y^2)(2z^2 + 2zw + 3w^2)\\ &= (2xz + xw + yz + 3yw)^2 + 5(xw - yz)^2\,.\end{aligned}$$

Try to find a similar identity for

$$(3x^2 + 2xy + 5y^2)(3z^2 + 2zw + 5w^2) = (\cdots)^2 + 14(\cdots)^2\,?$$

Q3) Test the following criteria for various primes p to check they do work.

- Let p be a prime, then

$$p = x^2 + 27y^2 \text{ if and only if } \begin{cases} p \equiv 1 \pmod 3\text{ , and} \\ 2 \equiv z^3 \pmod p \text{ has a solution} \end{cases}$$

- Let p be a prime, then

$$p = x^2 + 64y^2 \text{ if and only if } \begin{cases} p \equiv 1 \pmod 4\text{ , and} \\ 2 \equiv z^4 \pmod p \text{ has a solution} \end{cases}.$$

- Let $p \neq, 2, 7$ be a prime, then

$$p = x^2 + 14y^2 \text{ if and only if } \begin{cases} -14 \equiv z^2 \pmod p \text{ has a solution, and} \\ (z^2+1)^2 \equiv 8 \pmod p \text{ has a solution} \end{cases}.$$

A good selection is $p = 23, 31, 43, 73, 89, 109, 113, 127, 137, 151, 157$. A computer algebra system like Maple, Mathematica or Sage might be helpful. Or perhaps just a spreadsheet.