# Primes - Problem Sheet 4

## Properties of quadratic forms

Q1) Let $f(x_1, \ldots, x_n)$ be a quadratic form (with coefficients over some ring $R \supset \mathbb{Z}$). Show that

$$f \text{ is integral implies } 2f \text{ is } \textit{classically} \text{ integral.}$$

Q2) Suppose that $f(x_1, \ldots, x_n)$ is a non-primitive integral quadratic form. Show that $f(x_1, \ldots, x_n)$ can represent at most one prime.

Q3) Suppose $f(x, y) = ax^2 + bxy + cy^2$ is an integral binary quadratic form, with discriminant $D = b^2 - 4ac$.

i) Show that $f$ is indefinite if $D > 0$.

ii) Show that $f$ is positive (respectively negative) definite if $D < 0$ and $a > 0$ (respectively $a < 0$).

iii) What happens when $D = 0$? What happens if $D > 0$ is a perfect square? Hint: Complete the square!

Q4) Let $f(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form, of discriminant $D = b^2 - 4ac$. Show that $D \equiv 0, 1 \pmod 4$, and that every such $D$ occurs.

Q5) Show that $R$-equivalence is an equivalence relation on $n$-ary quadratic forms over $R$. Show

i) The form $f$ is equivalent to $f$,

ii) If $f$ is equivalent to $g$, then $g$ is equivalent to $f$, and

iii) If $f$ equivalent to $g$, and $g$ equivalent to $h$, then $f$ equivalent to $h$.
Check also for $\mathrm{SL}_n(\mathbb{Z})$-equivalence, when $R = \mathbb{Z}$.

Q6) Suppose $f$ and $g$ are $\mathrm{GL}_n(R)$-equivalent quadratic forms. Show

i) $\det(f)$ and $\det(g)$ differ by a square

$$\det(f) = \lambda^2 \det(g)\,,$$

for some $\lambda \neq 0 \in R^*$. How does $\lambda$ arise from the equivalence of $f$ to $g$?

ii) For $R = \mathbb{Z}$, conclude $\det(f) = \det(g)$, and explain why $\mathrm{GL}_n(\mathbb{Z})$-equivalent integral binary quadratic forms have the same discriminant.

Q7) Suppose $f$ and $g$ are $\mathrm{GL}_n(R)$-equivalent quadratic forms. Show

i) $f$ represents $r \in R$ if and only if $g$ represents $r \in R$.

ii) For $R = \mathbb{Z}$, $f$ represents $n \in \mathbb{Z}$ properly, if and only if $g$ represents $n \in \mathbb{Z}$ properly. Check also for $\mathrm{SL}_n(\mathbb{Z})$-equivalence.
Use this to show that

$$x^2 + 14y^2\ ,\ 2x^2 + 7y^2 \text{ and } 3x^2 + 2xy + 5y^2$$

are not $\mathrm{GL}_n(\mathbb{Z})$-equivalent.

Q8) Suppose $f$ and $g$ are integral $n$-ary quadratic forms. Then $2f$ and $2g$ are classically integral. Show that

$f$ is $\mathrm{GL}_n(\mathbb{Z})$-equivalent to $g$ if and only if $2f$ is $\mathrm{GL}_n(\mathbb{Z})$-equivalent to $2g$.

Check also for $\mathrm{SL}_n(\mathbb{Z})$-equivalence.

Q9) Suppose $f, g, h$ are integral quadratic forms. Suppose $f$ and $g$ are improperly equivalent, and $g$ and $h$ are improperly equivalent. Show that $f$ and $h$ are *properly* equivalent.