

Primes - Problem Sheet 8

Genus theory revisited

Q1) Let $p \equiv 1 \pmod{8}$ be prime.

i) Let $\mathcal{C}(-4p)$ be the class group of discriminant $D = -4p < 0$. Use genus theory to prove that

$$\mathcal{C}(-4p) \cong (\mathbb{Z}/2^a\mathbb{Z}) \times G,$$

where $\#G$ is odd, and $a \geq 1$. And hence $2 \mid h(-4p)$. Hint: recall the fundamental theorem for finitely generated abelian groups. How many elements of order 2 are in $\mathcal{C}(-4p)$?

ii) Use Gauss's definition of genus to show that

$$2x^2 + 2xy + ((p+1)/2)y^2$$

is in the principal genus. Hint: it is easier to use the Jacobi symbol, not the Legendre symbol.

iii) Use Theorem 8.4 to show $\mathcal{C}(-4p)$ has an element of order 4, hence conclude $4 \mid h(-4p)$.