Primes - Problem Sheet 9

Cubic reciprocity

Q1) With cubic reciprocity, we can handle another one of Euler's conjectures:

$$4p = x^2 + 243y^2 \iff \begin{cases} p \equiv 1 \pmod 3 \text{ and} \\ 3 \equiv a^3 \pmod p \end{cases}$$

Let $p \equiv 1 \pmod 3$ be prime.

i) Use the proof of $p = x^2 + 27y^2$ to show that

$$4p = a^2 + 27b^2$$

where we can take $a \equiv 1 \pmod 3$.

ii) Conclue that $\pi = (a + 3\sqrt{-3}b)/2$ is a primary prime of $\mathbb{Z}[\pi]$, and that $p = \pi\overline{\pi}$.

iii) For $\pi = (a + 3\sqrt{-3}b)/2$, show that the supplementary laws can be written as

$$\left(\frac{\omega}{\pi}\right)_3 = \omega^{2(a+2)/3}$$
$$\left(\frac{1-\omega}{\pi}\right)_3 = \omega^{(a+2)/3+b}$$

iv) Conclude $\left(\frac{3}{\pi}\right)_3 = \omega^{2b}$.

v) Use this to prove Euler's conjecture, above.

Modular forms

Q1) Recall that M_k denotes the space of weight k modular forms.

i) Show that M_k is a $\mathbb{C}$-vector space.

ii) If k is odd, show that $M_k = \{ 0 \}$. Hint: consider $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

iii) Let $f \in M_k$ and $g \in M_\ell$ be two modular forms. Show that fg is also a modular form, and that $fg \in M_{k+\ell}$.

Remark: Don't worry too much about the holomorphic at $i\infty$ condition!

Q2) Find a relation between E_4E_6 and E_{10}. Hence derive an identity for σ_9 as a 'convolution' of σ_3 and σ_5 of the form

$$\sigma_9(n) = a\sigma_5(n) + b\sigma_3(n) + c\sum_{i=1}^{n} \sigma_3(i)\sigma_5(n-i)\,.$$

(Here a, b, c are certain rational numbers you should find.)