Primes - Problem Sheet 9 - Solutions

Cubic reciprocity

Q1) With cubic reciprocity, we can handle another one of Euler's conjectures:

$$4p = x^2 + 243y^2 \iff \begin{cases} p \equiv 1 \pmod 3 \text{ and} \\ 3 \equiv a^3 \pmod p \end{cases}$$

Let $p \equiv 1 \pmod 3$ be prime.

i) Use the proof of $p = x^2 + 27y^2$ to show that

$$4p = a^2 + 27b^2$$

where we can take $a \equiv 1 \pmod 3$.

Solution: We have $4p = (2a-3b)^2 + 27b^2$, where $A = 2a - 3b$ Then $A \equiv 2a$. By taking $\pm A$, we get $A \equiv 1 \pmod 3$. (Take $+$, for $a \equiv 2 \pmod 3$ and $-$, for $a \equiv 1 \pmod 3$. We can't have $a \equiv 0 \pmod 3$, else $3 \mid p$.)

ii) Conclue that $\pi = (a + 3\sqrt{-3}b)/2$ is a primary prime of $\mathbb{Z}[\pi]$, and that $p = \pi\overline{\pi}$.

Solution: An element $\frac{x+y\sqrt{-3}}{2}$ is in $\mathbb{Z}[\omega]$ if and only if $x \equiv y \pmod 2$. We have $a - 3b \equiv a - b \pmod 2$. But by reducing $4p = a^2 + 27b^2$ modulo 2, we get $0 = a + b$, so yes $\pi \in \mathbb{Z}[\omega]$.

We easily see $\pi\overline{\pi} = p$, so π is prime: $N(\pi) = p$. Finally π is primary? We need to check $\pi \equiv \pm 1 \pmod 3$. But yes:

$$\pi = \frac{1}{2}(3a' + 1 + 3\sqrt{-3}b) = \frac{1}{2} \equiv 2^{-1} \equiv 2 \equiv -1 \pmod 3\,.$$

iii) For $\pi = (a + 3\sqrt{-3}b)/2$, show that the supplementary laws can be written as

$$\left(\frac{\omega}{\pi}\right)_3 = \omega^{2(a+2)/3}$$

$$\left(\frac{1-\omega}{\pi}\right)_3 = \omega^{(a+2)/3+b}$$

Solution: We need to write $\pi = -1 + 3m + 3n\omega$. We have

$$\pi = \frac{1}{2}(3a' + 3b + 1 - 3b + 3\sqrt{-3}b) = -1 + \frac{3}{2}(a' + b + 1) + 3b\underbrace{\frac{-1+\sqrt{-3}}{2}}_{\omega}$$

Then

$$\left(\frac{\omega}{\pi}\right) = \omega^{m+n} = \omega^{\frac{1}{2}(a'+b+1)+b} = \omega^{\frac{1}{2}(a'+3b+1)}$$

But $\frac{1}{2}(a' + 3b + 1) = \frac{1}{2}((a-1)/3 + 3b + 1) = \frac{1}{6}(a + 2 + 9b)$. Since $2^{-1} = 2 \pmod 3$, we get

$$\omega^{\frac{1}{2}(a'+3b+1)} = \omega^{\frac{2}{3}(a+2+9b)} = \omega^{\frac{2}{3}(a+2)+3\frac{b}{2}} = \omega^{2(a+2)/3}\,.$$

Similarly

$$\left(\frac{1-\omega}{\pi}\right)_3 = \omega^{2m} = \omega^{(a'+b+1)},$$

but

$$a' + b + 1 = \frac{1}{3}(a-1) + b + 1 = \frac{a+2}{3} + b.$$

iv) Conclude $\left(\frac{3}{\pi}\right)_3 = \omega^{2b}$.
Solution: This follows by writing $3 = -\omega^2(1-\omega)^2$. So

$$\left(\frac{3}{\pi}\right)_3 = \omega^{4(a+2)/3+2(a+2)/3+2b} = \omega^{2(a+2)+2b} = \omega^{2b},$$

since $a \equiv 1 \pmod 3$.

v) Use this to prove Euler's conjecture, above.
Solution: Suppose $4p = x^2 + 243y^2$. Then $p \equiv 1 \pmod 3$ by reducing modulo 3. Also we have

$$\pi = \frac{1}{2}(x + 9\sqrt{-3}y),$$

where π can be assumed to be a primary prime of $\mathbb{Z}[\omega]$. Then $b = 3y$, so that

$$\left(\frac{3}{\pi}\right)_3 = \omega^{2\times 3y} = 1$$

This means $3 \equiv a^3 \pmod 3$.
Conversely, suppose $p \equiv 1 \pmod 3$ and $3 = a^3 \pmod 3$. Then write $4p = a^2 + 27b^2$, with $a \equiv 1 \pmod 3$. We need to show $3 \mid b$. But since $3 \equiv a^3 \pmod 3$, we have $2b \equiv 0 \pmod 3$, whence $3 \mid b$. So

$$4p = x^2 + 27(3(b/3))^2 = x^2 + 243(b/3)^2.$$

Modular forms

Q1) Recall that M_k denotes the space of weight k modular forms.
i) Show that M_k is a $\mathbb{C}$-vector space.

ii) If k is odd, show that $M_k = \{0\}$. Hint: consider $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

iii) Let $f \in M_k$ and $g \in M_\ell$ be two modular forms. Show that fg is also a modular form, and that $fg \in M_{k+\ell}$.

Remark: Don't worry too much about the holomorphic at $i\infty$ condition!

Solution: For i) this is a straight-forward check. If f and g are weight k modular forms, then one sees $f+g$ transforms as a weight k modular form by factoring out $(cz+d)^k$.

For ii), let f be a weight k modular form. we have that $f(-I \cdot z) = (0z - 1)^k f(z) = (-1)^k f(z)$. So if k odd, then $f(z) = -f(z)$ for all z, so f is identically 0.

For iii), in fg, we have a factor $(cz+d)^k \times (cz+d)^\ell = (cz+d)^{k+\ell}$, so fg is a weight $k+\ell$ modular form.

Q2) Find a relation between E_4E_6 and E_{10}. Hence derive an identity for σ_9 as a 'convolution' of σ_3 and σ_5 of the form

$$\sigma_9(n) = a\sigma_5(n) + b\sigma_3(n) + c\sum_{i=1}^{n} \sigma_3(i)\sigma_5(n-i)\,.$$

(Here a, b, c are certain rational numbers you should find.)
Solution: Since $\dim M_10 = 1$, we have that $E_4E_6 = E_10$ since the first coefficient of both sides is 1. Comparing other coefficients gives

$$\sigma_9(n) = \frac{21}{11]}\sigma_5(n) - \frac{10}{11}\sigma_3(n) + \frac{5040}{11}\sum_{i=1}^{n} \sigma_3(i)\sigma_5(n-i)\,.$$