Report

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Name: Steven Charlton

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Supervisor: Dr. Herbert Gangl

1 Background Reading

My background reading has consisted of a number of different strands:

Operads: Primarily from the book *Algebraic Operads* by Loday and Vallette.
- An operad is an object which encodes the operations and relations of an algebra
- Captures the notion of composition of functions with multiple inputs
- Generalisations deal with functions having multiple inputs and outputs, functions where inputs and outputs are interchangeable, functions where outputs live in different spaces, where there are notions of trace and scalar product, ...
- Constructions and relations between algebras lift to morphisms between their operads
- Provides a way of unifying and generalising constructions on algebras

(Motivic) Multiple Zeta Values: *Multiple Zeta Values* Lectures by Brown and Gangl.
- Theory of Iterated Integrals, and relation to Multiple Zeta Values
- Betti and de Rham Fundamental Groups of $\mathbb{P}^1$ minus 3 points
- Motivic Multiple Zeta Values as a purely algebraic lifting of MZVs via the period map
- The Galois coaction on motivic MZVs, a Galois theory for some transcendental numbers
- Shuffle and stuffle product as the source of relations among MZVs
- Exceptional relations on double zeta values, arising from a connection to modular forms

R-deco Tree/Polygon Algebra: *Multiple Polylogarithms, Polygons, Trees and Algebraic Cycles* by Gangl, Goncharov and Levin.
- Admissible algebraic cycles, and the correspondence with polylogarithms
- The algebra of R-deco trees and R-deco polyogns, and their differentials
- The Forest Cycling map from R-deco trees to (not necessarily admissible) algebraic cycles, the restriction to $F^\times$-deco trees gives admissible cycles
- The bar construction on R-deco polygons, relation to Hopf algebra of iterated integrals
- Notions of enhanced R-deco trees and polygons to encode algebraic-topological cycles

Plus numerous papers and articles on these and closely related topics.

2 Summary of Research Work

Using Brown's motivic MZV framework, I have been able to re-establish the result by BBBL:

$$\zeta(\{2^m, 1, 2^m, 3\}^n, 2^m) = \text{rational} \cdot \pi^{(4m+4)n+2m}$$

The rational factor is known explicitly as $\frac{2(m+1)}{(2(m+1)(2n+1))!}$, but motivic MZVs don't seem able to see this.

From the motivic proof of this identity, it becomes clear why this π^{wt}-rational family of MZVs should generalise to a π^{wt}-rational family of sums of MZVs:

$$\sum_{\sigma \in S_{2n+1}} \zeta(2^{a_{\sigma(1)}}, 1, 2^{a_{\sigma(2)}}, 3, \ldots, 2^{a_{\sigma(2n-1)}}, 1, 2^{a_{\sigma(2n)}}, 3, 2^{a_{\sigma(2n+1)}}) \in \mathbb{Q}\pi^{\text{wt}}$$

where wt is the weight of the MZVs. Here one inserts arbitrary blocks of 2^{a_i} in the $2n+1$ gaps between 1s and 3s, and then symmetrises.

It is know explicitly that:

$$\sum_{a_1+\cdots+a_{2n+1}=M} \zeta(2^{a_1}, 1, 2^{a_2}, 3, \ldots, 2^{a_{2n-1}}, 1, 2^{a_{2n}}, 3, 2^{a_{2n+1}}) = \binom{M+2n}{M} \frac{\pi^{\mathrm{wt}}}{(2n+1)(\mathrm{wt}+1)!}$$

But the above result shows that this splits up into a sum of even smaller π^{wt}-rational pieces. Given an admissible composition $a_1 + \cdots + a_{2n+1} = N$, then any permutation of it is also admissible. So the terms of the sum can be combined into permutations of given compositions – these permutations sum to a rational multiple of π^{wt}.

These results hold because the motivic action factors through derivations D_{2k+1}, all of which are identically zero here. My proof that D_{2k+1} is zero relies on decomposing the binary strings encoding these MZVs into blocks in such a way that reversing sequences of blocks leaves the family of strings invariant. This reversal procedure sets up a pairing of the terms in D_{2k+1}, so that each pair cancels, giving $D_{2k+1} = 0$.

A similar type of cancellation, using reversing moves on a block decomposition of the corresponding binary strings, appears when proving results like:

$$\sum_{\sigma \in S_n} \zeta(2a_{\sigma(1)}, \ldots, 2a_{\sigma(n)}) \in \mathbb{Q}\pi^{\mathrm{wt}}$$

This suggest there may be some deeper pattern underlying these examples.

I have experimented a little bit with Gangl's R-deco polygon algebra and tree algebra. It is clear that in either algebra elements can be composed in an operadic fashion, and the compositions are related in the same way the algebras area. Using the forest cycling map, this transfers to a composition on algebraic cycles, but unfortunately the composition of two admissible cycles is not necessarily admissible. I don't yet know how this composition should be interpreted.

Using vertex-vertex arrows in place of vertex-side arrows leads to the definition of something very close to another differential on the polygon algebra. When computing ∂^2, most terms seem to cancel, apart from some pair of terms like $a \wedge (x \wedge y)$ and $a \wedge (\operatorname{op} x \wedge \operatorname{op} y)$, which might be said to be 'equivalent'. Here, by $\operatorname{op} x$, I mean the polygon x with opposite orientation, in some precise sense.

There seems to be a very strong pattern to the non-cancellation. Coupled with the fact that numerous other variations on the definition of such a differential have many more non-cancelling terms, this suggest to me that this is the right proto-definition , and should be looked at in more detail. Understanding why these terms do not cancel should help in modifying the definition to make it a fully fledged differential.

3 Future Plans

I'd like to further investigate the pairwise cancellation of the terms in D_{2k+1} in these examples. The similarity in how the cancellation arises, from a pairing of terms formed by reversing blocks in a decomposition of the binary strings encoding the MZVs, suggests these two are examples of a more general block decomposition cancellation methods.

Perhaps there are some rules governing how the binary strings must be decomposed in order for this cancellation to happen. Reversing blocks in the 'correct' decomposition of one of the binary strings generates the other strings in the identity. Applying this to other block decompositions would then produce more identities.

So far I have only really looked at cases where the terms in D_{2k+1} cancel pairwise. I haven't yet made much use of the shuffle relations, which the iterated integrals satisfy, to cancel the terms.

I have incorporated these shuffle relations into some small programs, searching for identities. Currently they are unwieldy to work with, and so the program takes far too long to produce any results. Spending some time making this more efficient may throw up some other interesting identities. There is enough hint of structure and symmetry in an example like:

$$\zeta(21212 + 12122 + 11222 + 21122 + 2213) \in \mathbb{Q}\pi^8$$

to support this.

The cancellation of D_{2k+1} that happens in these examples may have some bearing on the rules for block decomposition.

The cyclic insertion conjecture proposes that the sum:

$$\sum_{\sigma \in \langle (1,2,\ldots,2n+1) \rangle < S_{2n+1}} \zeta(2^{a_{\sigma(1)}}, 1, 2^{a_{\sigma(2)}}, 3, \ldots, 2^{a_{\sigma(2n-1)}}, 1, 2^{a_{\sigma(2n)}}, 3, 2^{a_{\sigma(2n+1)}}) = \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

and that this is independent of the sum of the a_i and of the number of terms.

Apart from the case $A_3 \cong C_3 < S_3$, which can be handled by duality as half of the full S_3 sum, and the case where only one block of 2s is inserted, meaning the S_{2n+1} sum is a multiple of the C_{2n+1} sum, even the fact that these are π^{wt}-rational doesn't fall under the purview of the result above. The proof I find for that result relies on setting up a pairwise cancellation of terms using reversal of paths. Using the shuffle relations iterated integrals satisfy, and the fact that the left tensor factor of terms in D_{2k+} lies in the space of indecomposables – iterated integrals modulo non-trivial products – I imagine it should be possible to also show this sum is a rational multiple of π^{wt}.

Although Brown's motivic MZV framework does not seem capable to explicitly finding the rational factor, Gangl suggests that some shadow of this factor may be cast in the motivic structure. Is it possible to see the ratio of MZVs is a certain rational factor from the motivic level, without having their explicit value? He has suggested viewing $\zeta^{\mathfrak{m}}(2)$ as some 24-torsion object, in the hopes of extracting more information. In establishing that certain MZVs are rational multiples of $\zeta^{\mathfrak{m}}(n)$, only the odd derivations D_{2k+1} are used. Do the even degree ones reveal any more structure?

I intend to look also at how/if these ideas extend to Euler-Zagier sums and other MZV-like sums associated to higher roots of unity. It has been suggested that Euler-Zagier sums are the 'correct' choice in that their algebraic structure is better behaved, they have a nicer basis and so on.

The operadic structure on the R-deco polygon, and R-deco tree algebra, and what this means for the algebraic cycles deserves consideration. Admissible algebraic cycles are related very precisely to polylogarithms; a composition on algebraic cycles should give rise to something on the polylogarithm side. Unfortunately the composition of admissible algebraic cycles is not necessarily admissible. When this composition is admissible, what does it say on the polylogarithm side?

Can I modify the definition of the VV-differential in order to avoid the remaining terms in ∂^2? What does the existence of this differential mean through for the R-deco polygon algebra, R-deco tree algebra, algebraic cycles, and polylogarithms sides?