Report

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1 Background Reading

My background reading has primarily consisted of:

Dan's Hyperlogrithms: from [2]

Dan introduces a slight variant of the usual hyperlogarithm by using the change of variables $z \mapsto 1/z$ on $\mathbb{P}^1(\mathbb{C})$ to allow extra types of differential forms. This leads to the function $H(a_0 \mid a_1, \ldots, a_n \mathbin{/\!/} x \mid a_{n+1})$, which reduces to the usual hyperlogarithm when $x = \infty$.

With the extra flexibility of a new type of variable x gives, Dan describes a method to write a hyperlogarithm in n variables as the sum of hyperlogarithms in $\leq n-2$ variables, modulo products. This is built up in stages by first relating $H(a_0 \mid a_1, \ldots, a_n \mathbin{/\!/} x \mid a_{n+1}$ to $H(a_0 \mid a_1, \ldots, x, \ldots, a_n \mathbin{/\!/} a_i \mid a_{n+1})$ (obtained by swapping $a_i \leftrightarrow x$) and using this to build up to an arbitrary permutation of $a_1, \ldots, a_n$.

By combining these Dan can write each terms of shuffle product $H(a_0 \mid a_1 a_2 \sqcup\!\sqcup a_3 \ldots a_n \mid a_{n+1})$ as $H(a_0 \mid a_1, \ldots, a_n \mid a_{n+1})$ and sums of hyperlogarithms in $\leq n-2$ variables. And this gives the desired expression.

Symbols of Polylogarithms: from [8] and [3]

The symbol of a polylogarithm (or more generally a transcendental function defined as the iterated integral of certain differential forms) is a kind of algebraic invariant attached to the function. It captures the differential properties of the function without worrying about the analytic details and the multivalued-ness.

Given a function $F = \int_0^z \mathrm{d}\log(f_1) \circ \cdots \circ \mathrm{d}\log(f_n)$, all we need to know to recover F is the differential forms and their order. The object $f_1(z) \otimes \cdots \otimes f_n(z)$ tells us this, and this is essentially the symbol of F. It is possible to express multiple polylogarithms as a sum of terms of this form, to write their symbol

The algebraic operations of addition and multiplication on polylogarithms map to addition and shuffle product of symbols, so that functional equations between polylogrithms become algebraic equations on symbols. This gives a way to check whether candidate identities could be correct. With careful use one can even derive functional equations by integrating the symbol.

Goncharov's Geometry of Trilogarithms: from [6] and [5]

The dilogarithm can be interpreted geometrically as a configuration of 4 points on $\mathbb{P}^1(\mathbb{C})$, with the 5-term functional equation coming by removing each point of a configuration of 5 points in turn.

Goncharov gives a similar interpretation for the trilogarithm, as a special configuration of 6 points in $\mathbb{P}^2(\mathbb{C})$. Functional equations arise by removing each point of a configuration of 7 points in turn. From this he can derive a new, general, functional equation for the trilogarithm.

This geometric view point allowed Goncharov to explicitly write down the regulator of K-theory, and prove Zagier's conjecture in the case $n = 3$.

Aomoto Polylogarithms and Double Scissors Congruence Groups: from [7] and [1]

Aomoto polylogarithms give another geometric viewpoint and generalisation of polylogarithms. One can write a (multiple) polylogarithm as an iterated integral, which one can then view as an integral over a simplex in n-space. One can also naturally view the differential form as being associated to another simplex.

Aomoto is a vast generalisation of this in terms of what simplices are used. Any pair of admissible simplices (L, M) in $\mathbb{P}^n$ can be used to define an integration region Δ_M and a canonical differential form ω_L to integrate on this region. Aomoto polylogarithms satisfy the properties of non-degeneracy, skew-symmetry, additivity in L and in M, and projective invariance. These algebraic properties can be abstracted out to give the definition of the double scissors congruence groups over an arbitrary field, giving some notion of an abstract (Aomoto) polylogarithm.

These double scissors congruence groups, and whether they fit together into a Hopf algebra, have some deep connections with questions and conjectures about K-theory.

2 Summary of Research Work

I have clarified where my symmetric insertion result

$$\sum_{\sigma \in S_{2n+1}} \zeta(2^{a_{\sigma(1)}}, 1, 2^{a_{\sigma(2)}}, 3, \ldots, 2^{a_{\sigma(2n-1)}}, 1, 2^{a_{\sigma(2n)}}, 3, 2^{a_{\sigma(2n+1)}}) \in \mathbb{Q}\pi^{\mathrm{wt}}$$

fits in the context of known and conjectured results. Borwein, Bradley, and Broadhurst only have a *conjectural* evaluation of $\zeta(\{2^m, 1, 2^m, 3\}^n, 2^m)$ as an explicit rational mutiple of π^{wt}. Symmetric insertion shows that it is indeed *some* rational multiple of π^{wt}.

I have written this result up and uploaded it to the arXiv `http://arxiv.org/abs/1306.6775`.

I have also spent some time looking at certain properties of the level filtration of the Hoffman basis of multiple zeta values. Brown's development of Motivic MZVs introduces the filtration $H_i = \{\ q\zeta(\text{2s and 3s}) \mid q \in \mathbb{Q} \text{ and } \leq i \text{ 3s }\ \}$, and as pointed out by Terasoma leads to $H_i H_j \subset H_{i+j}$.

Using the `zeta_proc` routines for Maple to decompose some products $\zeta(w)\zeta(w')$ into the Hoffman basis, shows up some interesting features. For example:

$$\zeta_{2223}\zeta_2 = \frac{10265472}{2555171}\zeta_{32222} - \frac{9915552}{2555171}\zeta_{23222} - \frac{10397160}{2555171}\zeta_{22322} - \frac{5838895}{2555171}\zeta_{22232} + \frac{37505700}{2555171}\zeta_{22223}$$
$$\zeta_{22}\zeta_{322} = \frac{370811826}{12775855}\zeta_{32222} + \frac{520697499}{12775855}\zeta_{23222} + \frac{85939563}{2555171}\zeta_{22322} + \frac{556488492}{12775855}\zeta_{22232} + \frac{6514704}{2555171}\zeta_{22223}$$

It is curious that the same large (prime) number 2555171, or a small multiple of it $12775855 = 5 \times 2555171$, appears in all the denominators of both decompositions. A similar observation holds in other cases too.

For level 1 times level 0, I can explain this observation by giving an explicit formula for the decomposition of $\zeta(2^a, 3, 2^b)\zeta(2^c)$ using Zagier's matrices from [9]. The inverse of the $(a+b+c+1) \times (a+b+c+1)$ matrix is the source of the denominators. I unsure how to explain it for higher levels.

As mentioned above, Dan describes a general method for writing hyperlogarithms as sums of hyperlogs in fewer variables, modulo products. At the end of the paper he gives an expression for $H(a \mid b, c, d, e \mid f)$ in terms of $I_{31}(x, y)$ and polylogarithms $I_4(x)$. Unfortunately his expression is incorrect as the symbols do not agree.

I have spent time understanding Dan's method with goal of being able to correct his expression. Unfortunately Dan makes use of some unspecified identities to write $I_{13}(x, y)$ and $I_{22}(x, y)$ in terms of $I_{31}(x, y)$, so I have not been able to correct *his* expression. However I have successfully implemented his method to produce a correct identity for $H(a \mid b, c, d, e \mid f)$ in terms of I_{13}, I_{22}, I_{31} and I_4. Using some identities relating I_{22} and I_{13} with I_{31} provided by Gangl, I can rewrite it purely in terms of I_{31} and I_4 to the same end.

Dan neglects products throughout his method, so the identities he obtains are only true modulo products. Whilst working through the steps of his method, I have also restored the neglected product terms, and so can give unconditionally correct identities. For example, using the shorthand notation $[x,y]_{11} = I_{11}(x,y)$ and so on, along with $abc = \text{cr}(a,b,c,\infty) = \frac{a-c}{b-c}$ and $abcd = \text{cr}(a,b,c,d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$, I have:

$$
\begin{aligned}
&H(a \mid b,c,d \mid e) + H(a \mid c,b,d \mid e) = \\
&+\{-[dab]_3 + [deb]_3 - [dab]_1[bea,bea]_{11} - [bea]_1[dab]_2\} - [dea]_1[bea,bea]_{11} + [bea]_1[bea,dea]_{11} \\
&-2\{-[cab]_3 + [ceb]_3 - [cab]_1[bea,bea]_{11} - [bea]_1[cab]_2\} + [bea]_1[cea,bea]_{11} - [bea,bea,bea]_{111} \\
&-\{-[cbda]_3 + [cbde]_3 - [cbda]_1[abce,abce]_{11} - [abce]_1[cbda]_2\} + [abde]_1[abce,abce]_{11} - [abce]_1[abce,abde]_{11} \\
&-\{-[bca]_3 + [bce]_3 - [bca]_1[abce,abce]_{11} - [abce]_1[bca]_2\} + [abce,abce,abce]_{111} \\
&+\{-[adc]_3 + [edc]_3 - [adc]_1[cae,cae]_{11} - [cae]_1[adc]_2\} + [cae]_1[cae,acde]_{11} - [acde]_1[cae,cae]_{11} \\
&-2\{-[abc]_3 + [ebc]_3 - [abc]_1[cae,cae]_{11} - [cae]_1[abc]_2\} + [cae]_1[acbe,cae]_{11} - [cae,cae,cae]_{111}
\end{aligned}
$$

One observation here is that the overall expression must be symmetric under $b \leftrightarrow c$, but this is not readily apparent on the right hand side. In fact under this symmetry line 3 plus line 4 symmetric, and lines 1 plus 2 map into lines 5 plus 6, which is made obvious during the derivation of the result.

3 Future Plans

One current disadvantage of my implementation of Dan's method in Mathematica is that an unnecessary amount of structure present in the method is lost during the computations. It is an fundamental property of Dan's method that the number of terms grows rapidly as the weight increases. Reworking the implementation to keep more of the structure and to keep track of where each term originates would better help highlight any patterns. Even more so when including the product terms as well. Once done I will be able to calculate the exact expression for $H(a \mid b,c,d,e \mid f)$, without neglecting products.

Gangl has suggested that writing $H(a \mid b,c,d,e \mid f)$ entirely in terms of I_{31} and polylogs only may not be the best choice, and that I_{22} may be a more sensible candidate due to its symmetries. One wants to get rid of any multiple polylogarithm terms with index 1.

Often it is the case that symmetrising an expression leads to simplification, and more recognisable structure and patterns. I should try to symmetrise the various expressions I have for $H(a \mid b,c,d \mid e)$ and $H(a \mid b,c,d,e \mid f)$ to try and tease out some more structure.

Having various geometric ways of views polylogarithms, it makes sense to ask whether there is any geometric significance or interpretation of the weight 4 expression. For example in the Aomoto settings it is know that every simplex in $\mathbb{P}^2$ can be decomposed into dilogarithmic simplices, and every simplex in $\mathbb{P}^3$ into trilogarithmic simplices. The appropriate question for $\mathbb{P}^4$ is can every simplex be decomposed into tetralogarithmic simplices and $\text{Li}_{2,2}$-simplices. Maybe the expressions for $H(a \mid b,c,d,e \mid f)$ can shed some light here?

Similarly, is there any interpretation in Goncharov's trilog configurations for the weight 3 expressions?

Since we can compute the symbol of the trilogarithm, and Goncharov gives a geometric interpretation of the trilog as a particular configuration of 6 points in $\mathbb{P}^2$, (how) can the symbol be extended to any configuration of points? A geometric interpretation of the weight 3 expressions may be useful here.

For the usual iterated integrals, there are recursive methods to compute the symbol, using that fact that $\mathrm{d}\,F$ can be expressed as lower weight iterated integrals, to build up the symbol up iteratively. In [4], Goncharov gives a formula for d Aomoto (as a special case of the differential of the period of a variation of Hodge-Tate structures), in terms of lower weight Aomoto polylogarithms. The same iterative procedure should allow me to build up the symbol of an Aomoto polylogarithm.

There are explicit formulae for the general Aomoto dilog and trilog in terms of classical polylogarithms. I would be able to use these to verify my computation of the Aomoto symbol for weights 2 and 3, by checking that it agrees with the symbol of the classical polylog expressions.

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