Quadratic forms, theta series, η-products and Ramanujan-τ congruences

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I am interested in studying which primes are represented by certain quadratic forms. In particular, giving conditions for those forms which cannot be handled by class field theory (as in [Cox89]). For class number $h(D) \leq 4$ or $h(D) = 6$, class field theory is sufficient to completely characterise the primes represented by *any* class of quadratic forms. How can we go beyond this?

1. Polynomial congruences are insufficient

I try to indicate what I mean by this. Recall there is an isomorphism from the form class group $\mathcal{C}(D)$ of discriminant D to the (narrow) ideal class group $\mathcal{C}(\mathcal{O})$ of the order $\mathcal{O} = \mathbb{Z}[\sqrt{D}] \subset \mathcal{O}_K$, where $K = \mathbb{Q}(\sqrt{D})$. Using the ring class field $K_\mathcal{O}$ to study when the prime $(p)_\mathbb{Z}$ of $\mathbb{Z}$ splits into principal ideals in $\mathcal{O} = \mathbb{Z}[\sqrt{D}]$, we always obtain a condition of the following form: for $p \mid \operatorname{disc} 2D f_D(t)$

$$p = x^2 + ny^2 \iff \begin{cases} (-D/p) = 1, \text{ and} \\ f_D(t) \equiv 0 \pmod{p} \text{ has a root,} \end{cases}$$

Here $f_D(t)$ the polynomial of degree $h(D)$, which defines the ring class field $K_\mathcal{O}$ of the order $\mathcal{O} = \mathbb{Z}[\sqrt{D}]$ as an extension of K. (The polynomial $f_D(t)$ can be taken to have $\mathbb{Z}$-cooefficients.)

Since $\operatorname{Gal}(K_\mathcal{O}/K) \cong \mathcal{C}(\mathcal{O})$, intermediate fields of the ring class field $K_\mathcal{O}/K$ correspond to subgroups $G \subset \mathcal{C}(\mathcal{O})$ of the (narrow) ideal class group. We can use these intermediate fields to detect when a prime $(p)_\mathbb{Z}$ splits into ideals which lie in this subgroup. Equivalently, this detects when a prime p can be represented by some quadratic form lying in a corresponding subgroup $G' < \mathcal{C}(D) \cong \mathcal{C}(\mathcal{O})$.

For class number $h(D) \leq 4$, and $h(D) = 6$, one can exploit the structure of the form class group, and the fact that the Gauss-composition-inverse quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ and $Q^{-1}(x, y) = ax^2 - bxy + cy^2$ obviously represent the same primes. Since forms which represent the same primes are $\mathrm{GL}_2(\mathbb{Z})$-equivalent, one can use inclusion-exclusion to give explicit criteria describing the primes represented by any of the forms of discriminant D.

For example, consider class number $h(D) = 6$, where necessarily $\mathcal{C}(D) \cong \mathbb{Z}/6\mathbb{Z} = \langle Q | Q^6 = 1 \rangle$. We can consider the following subgroups $G_1 = \{e\}$, $G_2 = \{e, Q^3\}$, $G_3 = \{e, Q^2, Q^4\}$ and $G_6 = \mathbb{Z}/6\mathbb{Z}$. Then

$$\begin{aligned} \{e\} &= G_1 \\ \{Q^3\} &= G_2 \setminus G_1 \\ \{Q^2, Q^4\} &= G_3 \setminus G_1 \\ \{Q^5, Q^1\} &= G_6 \setminus (G_1 \cup G_3 \cup G_2) \end{aligned}$$

Whence one obtains the criterion, for p not dividing $2D$ times the discriminants of the polynomials $f_i(t)$.

$$p = e(x, y) \iff \begin{cases} (-D/p) = 1, \text{ and} \\ f_1(t) \equiv 0 \pmod{p} \text{ has a root.} \end{cases}$$

$$p = Q^3(x, y) \iff \begin{cases} (-D/p) = 1, \text{ and} \\ f_2(t) \equiv 0 \pmod{p} \text{ has a root, and} \\ f_1(t) \equiv 0 \pmod{p} \text{ no roots.} \end{cases}$$

$$p = Q^2(x,y) \iff \begin{cases} (-D/p) = 1, \text{ and} \\ f_3(t) \equiv 0 \pmod p \text{ has a root, and} \\ f_1(t) \equiv 0 \pmod p \text{ no roots.} \end{cases}$$

$$p = Q^2(x,y) \iff \begin{cases} (-D/p) = 1, \text{ and} \\ f_3(t) \equiv 0 \pmod p \text{ no roots, and} \\ f_2(t) \equiv 0 \pmod p \text{ no roots, and} \\ f_1(t) \equiv 0 \pmod p \text{ no roots.} \end{cases}$$

Here $f_i(t)$ is the polynomial defining the intermediate fixed field $K \subset K_{\mathcal{O}}^{G_i} \subset K_{\mathcal{O}}$ of the ring class field $K_{\mathcal{O}}/K$, the one which corresponds to the subgroup G_i of the ideal/form class group $\mathcal{C}(D)$.

This inclusion-exclusion principle works to give criteria for all forms, if and only if every element of the class group has order $1, 2, 3, 4$ or 6. If there is an element of order 5, or order ≥ 7, then we cannot distinguish the $\phi(5) = 4$ or $\phi(\geq 7) \geq 4$ primitive generators – and these forms do *not* represent the same primes.

The first time where we run into difficulties it at class number $h(D) = 5$, where (as indicated above) we cannot distinguish the forms Q, Q^2, Q^3 and Q^4 using subgroups of the class group/intermediate fields of the ring class field. Therefore, we cannot apply our strategy from above to obtain a 'boolean system' of polynomial congruences characterises the primes represented by Q. But by itself this does not mean that such a system does not exist, only we do not have the necessary techniques to find it.

To focus on a a concrete case, take $D = -47$ where we have the following representatives of the 5 classes of binary quadratic forms

$$\underbrace{x^2 + xy + 12y^2}_{e(x,y)}, \quad \underbrace{2x^2 + xy + 6y^2}_{Q(x,y)}, \quad \underbrace{3x^2 - xy + 4y^2}_{Q^2(x,y)},$$
$$\underbrace{3x^2 + xy + 4y^2}_{Q^3(x,y)}, \quad \underbrace{2x^2 - xy + 6y^2}_{Q^4(x,y)}$$

Question 1. The quadratic form $Q(x,y) = 2x^2 + xy + 6y^2$ represents the following list of primes

$$p = 2, 7, 53, 59, 61, 89, 97, 131, 157, 173, 263, 283, 331, 337, 353, 379, 431, 479, 491, \ldots$$

Are these primes characterised by a boolean system of polynomial congruences? How does one find this system, or prove that it does cannot exist?

Musing 2. If one could prove that the polynomials in such a system had to define intermediate fields of the ring class field, one could say that such a system does not exist since the degree 5 Abelian extension $K_{\mathbb{Z}[\frac{1+\sqrt{-47}}{2}]}/\mathbb{Q}(\sqrt{-47})$ has no intermediate fields. But currently, I see no reason why such polynomials would have to have a connection to class fields of any sort.

2. Quadratic forms of discriminant -47

Assuming, then, that no polynomials are sufficient to characterise those primes, we are compelled to ask how they *can* be characterised (in some explicit, *non-tautological* way!). For this we can consider the Θ-series of the quadratic forms, and use results from the theory of modular forms.

I focus first on the specific case $D = -47$, then ask about generalities. From the 5 quadratic forms above, we obtain only the following 3 distinct Θ-series

$$\Theta_0 = \Theta_{x^2+xy+12y^2}(z) = 1 + 2q + 2q^4 + 2q^9 + 4q^{12} + \cdots$$
$$\Theta_1 = \Theta_{2x^2+xy+6y^2}(z) = 1 + 2q^2 + 2q^6 + 2q^7 + 2q^8 + 2q^9 + 2q^{12} + \cdots$$
$$\Theta_2 = \Theta_{3x^2+xy+4y^2}(z) = 1 + 2q^3 + 2q^4 + 2q^6 + 2q^8 + 2q^{12} + \cdots$$

By the 'standard' theory, these Θ-series are modular forms forms of weight 1, for $\Gamma_0(|D|) = \Gamma_0(47)$, with character $\varepsilon_{-47} = (-47/\cdot)$. One obvious modular form in this space (actually in the subspace of cusp forms) is the η-product

$$\eta(z)\eta(47z) = q^2 \prod_{n=1}^{\infty}(1-q^n)(1-q^{47n})\,.$$

It turns out that

$$\frac{1}{2}(\Theta_1 - \Theta_2) = \eta(z)\eta(47z) =: \sum_{n=1}^{\infty} a_n q^n\,,$$

and this difference of Θ-series already holds enough information for us to write down precise, explicit criterion for the 5 forms. Writing a_n for the coefficient of q^n in the q-expansion of $\eta(z)\eta(47z)$, we obtain the following

$$p = x^2 + xy + 12y^2 \iff \begin{cases} (-47/p) = 1, \text{ and} \\ a_p = 0 \end{cases}$$

$$p = 2x^2 \pm xy + 6y^2 \iff \begin{cases} (-47/p) = 1, \text{ and} \\ a_p = 1 \end{cases}$$

$$p = 3x^2 \pm xy + 4y^2 \iff \begin{cases} (-47/p) = 1, \text{ and} \\ a_p = -1 \end{cases}$$

This gives us criteria in terms of the coefficient of 'known' modular forms. In fact we can make this somewhat more explicit in the following way. Notice that modulo 47 (prime), we have

$$\begin{aligned} \eta(z)\eta(47z) &= q^2 \prod_{n=1}^{\infty}(1-q^n)(1-q^{47n}) \\ &\equiv q^2 \prod_{n=1}^{\infty}(1-q^n)(1-q^n)^{47} \\ &= \Delta(z)^2 \pmod{47} \end{aligned}$$

In particular $a_n \equiv \sum_{i+j=n} \tau(i)\tau(j) \pmod{47}$, where τ is the Ramanujan-τ functon, the coefficient of q^n in the q-expansion of $\Delta(z)$. So we can replace the a_p criteria above with explicit criteria involving a congruence on a convolution of the Ramanujan-τ function.

$$p = x^2 + xy + 12y^2 \iff \begin{cases} (-47/p) = 1, \text{ and} \\ \sum_{i+j=p} \tau(i)\tau(j) \equiv 0 \pmod{47} \end{cases}$$

$$p = 2x^2 \pm xy + 6y^2 \iff \begin{cases} (-47/p) = 1, \text{ and} \\ \sum_{i+j=p} \tau(i)\tau(j) \equiv 1 \pmod{47} \end{cases}$$

$$p = 3x^2 \pm xy + 4y^2 \iff \begin{cases} (-47/p) = 1, \text{ and} \\ \sum_{i+j=p} \tau(i)\tau(j) \equiv -1 \pmod{47} \end{cases}$$

Alternatively, one can read this as a congruence for a Ramanujan-τ convolution.

Theorem 1. *For any prime p, the convolution-square of the Ramanujan-τ function satisfies the following congruences modulo 47*

$$\sum_{i+j=p} \tau(i)\tau(j) \equiv \begin{cases} 0 & \text{if } (-47/p) = -1 \\ 0 & \text{if } (-47/p) = 0 \\ 0 & \text{if } (-47/p) = 1 \text{ and } p = x^2 \pm xy + 12y^2 \\ 1 & \text{if } (-47/p) = 1 \text{ and } p = 2x^2 \pm xy + 6y^2 \\ -1 & \text{if } (-47/p) = 1 \text{ and } p = 3x^2 \pm xy + 4y^2 \end{cases} \pmod{47}$$

It appears that for any other prime modulus r, the set

$$\left\{ \sum_{i+j=p} \tau(i)\tau(j) \pmod{r} \,\middle|\, p \text{ prime} \right\}$$

exhausts all residue classes in $\mathbb{Z}/r\mathbb{Z}$.

This is reminiscent of Wilton's congruence [Wil30] for $\tau(p)$ modulo 23, which is obtained by considering quadratic forms of discriminant -23. Swinnerton-Deyer also observes the further congruence that

$$\tau(p) \equiv \sigma_{11}(p) \pmod{23^2}$$

if $p = x^2 + 23y^2$. Serre [Ser67] provides a proof/interpretation of this, though I do not yet fully understand the details of this proof.

Conjecture 2. *It appears that one has an analogue of this modulo 47, namely for $p \neq 47$:*

$$p = x^2 + 47y^2 \implies \sum\nolimits_{i+j=p} \tau(i)\tau(j) \equiv 0 \pmod{47^2},$$

but it is not yet clear to me what the congruence is modulo 47^3.

Question 3. Is there any literature about the above theorem? Does the conjectural refinement that for $p \neq 47$:

$$p = x^2 + 47y^2 \implies \sum\nolimits_{i+j=p} \tau(i)\tau(j) \equiv 0 \pmod{47^2},$$

hold? If so what does the congruence modulo 47^3 look like?

The standard theory tells us that

$$\Theta_0 + 2\Theta_1 + 2\Theta_2 = \frac{5}{2} + \sum_{n=1}^{\infty} \Bigg(\sum_{d \mid n} \left(\frac{-47}{d} \right) \Bigg) q^n$$

(arising from the trivial character of $\mathbb{Z}/5\mathbb{Z}$) is an Eisenstein series. We also have that

$$S_1 = \Theta_0 + (\zeta_5 + \zeta_5^4)\Theta_1 + (\zeta_5^2 + \zeta_5^3)\Theta_2 = \Theta_0 + \tfrac{-1+\sqrt{5}}{2}\Theta_1 + \tfrac{-1-\sqrt{5}}{2}\Theta_2$$
$$S_2 = \Theta_0 + (\zeta_5^2 + \zeta_5^3)\Theta_1 + (\zeta_5 + \zeta_5^4)\Theta_2 = \Theta_0 + \tfrac{-1-\sqrt{5}}{2}\Theta_1 + \tfrac{-1+\sqrt{5}}{2}\Theta_2$$

(arising from the non-trivial characters of $\mathbb{Z}/5\mathbb{Z}$) are Hecke eigenforms for $S_1(\Gamma_0(47), \varepsilon_{-47})$.

We already know one basis element of $S_1(\Gamma_0(47), \varepsilon_{-47})$, in terms of η-products (or other 'familiar' modular forms). This space is 2 dimensional, so we only need to find a second basis vector. The method from [Kil08] seems to apply, to say every modular form for $S_1(\Gamma_0(47), \varepsilon_{-47})$ is a rational function in Dedekind η's. Namely, if we consider

$$f_i(z) := \frac{S_i(\eta(z)\eta(47z))^{11}}{\Delta(z)},$$

this is modular of weight 0, for $\Gamma_0(47)$, with no character. Therefore by [Mil17] Theorem 6.1, one can say that f_i is a rational function of $j(z), j(47z)$, for the Klein j-invariant. This can then be expressed in terms of η since the Eisenstein series E_4 and E_6 have known expressions in terms of η. Unfortunately, it seems rather difficult to do this explicitly, particularly since we are looking for a *rational* function in η, and not just a sum of η-quotients.

Question 4. How can one find an explicit expression for the second basis element of $S_1(\Gamma_0(47), \varepsilon_{-47})$ as a rational function of η? Are there any computational techniques that make this easier to approach.

Alternatively, are these other 'familiar' modular forms which can serve as this second basis vector? (Ideally, I want to find, in principal, an expression for the coefficients of the 3 Θ-series above. So I want to relate the Θ-series to some *other* modular form, which has a more explicit/well-known Fourier coefficient.)

3. Quadratic forms of discriminant -79

The next time that class number $h(D) = 5$ occurs, is for $D = -79$. Unfortunately, the same analysis as for $D = -47$ now runs into difficulties much earlier. Since $1 + |D| = 80$ is not divisible by 24, the η-product $\eta(z)\eta(79z)$ is not a modular form for $S_1(\Gamma_0(79), \varepsilon_{-79})$, so we can't even run the method indicated above, so how to find an explicit basis for $S_1(\Gamma_0(79), \varepsilon_{-79})$.

Question 5. Can one find a basis for $S_1(\Gamma_0(79), \varepsilon_{-79})$ in terms of rational functions of η? Alternatively, can any other 'familiar' modular forms provide such a basis?

4. Quadratic forms of discriminant -71

We can play the same game as above to obtain some results for discriminant -71, where $h(-71) = 7$. We have the following 7 classes of quadratic forms

$$\underbrace{x^2+xy+18y^2}_{e(x,y)},\quad \underbrace{2x^2+xy+9y^2}_{Q(x,y)},\quad \underbrace{4x^2-3xy+5y^2}_{Q^2(x,y)},\quad \underbrace{3x^2+xy+6y^2}_{Q^3(x,y)},$$
$$\underbrace{3x^2-xy+6y^2}_{Q^4(x,y)},\quad \underbrace{4x^2+3xy+5y^2}_{Q^5(x,y)},\quad \underbrace{2x^2-xy+9y^2}_{Q^6(x,y)}$$

The associated Θ-series are

$$\begin{aligned}\Theta_0 &= 1+2q+2q^4+2q^9+\cdots\\ \Theta_1 &= 1+2q^2+2q^8+2q^9+2q^{10}+2q^{12}+\cdots\\ \Theta_2 &= 1+2q^4+2q^5+2q^6+2q^{12}+\cdots\\ \Theta_3 &= 1+2q^3+2q^6+2q^8+2q^{10}+2q^{12}+\cdots\end{aligned}$$

We have

$$\frac{1}{2}(\Theta_3-\Theta_2) = \eta(z)\eta(71z)\,.$$

We can distinguish $Q^2(x,y)$ and $Q^3(x,y)$ from the remaining forms $e(x,y)$ and $Q^1(x,y)$ using the coefficient b_p of the q-expansion of the modular form $\eta(z)\eta(71z)$. We can then distinguish $e(x,y)$ and $Q^1(x,y)$ using the Hilbert class field condition. So we obtain the following criterion

$$p = x^2+xy+18y^2 \iff \begin{cases}(-71/p)=1, \text{ and}\\ t^7-t^6+2t^5-2t^4+t^3+2t^2-3t+1\equiv 0 \pmod p \text{ has root}\end{cases}$$

$$p = 2x^2\pm xy+9y^2 \iff \begin{cases}(-71/p)=1, \text{ and}\\ b_p=0, \text{ and}\\ t^7-t^6+2t^5-2t^4+t^3+2t^2-3t+1\equiv 0 \pmod p \text{ no root}\end{cases}$$

$$p = 4x^2\pm 3xy+5y^2 \iff \begin{cases}(-71/p)=1, \text{ and}\\ b_p=-1\end{cases}$$

$$p = 3x^2\pm xy+6y^2 \iff \begin{cases}(-71/p)=1, \text{ and}\\ b_p=1\end{cases}$$

Considering $\eta(z)\eta(71z)$ modulo 71 leads to the result that $\eta(z)\eta(71z)\equiv\Delta^3(z) \pmod{71}$, and so

$$b_n \equiv \sum_{i+j+k=n}\tau(i)\tau(j)\tau(k) \pmod{71}\,.$$

We can replace $b_p=-1,0,1$ with the corresponding congruence to obtain an explicit condition on when p is represented by the corresponding quadratic forms, by using the Ramanujan-τ function. Alternatively, we obtain a congruence on the triple-convolution of Ramanujan-τ.

Theorem 3. *For any prime p, the convolution-cube of the Ramanujan-τ function satisfies the following congruence modulo 71.*

$$\sum_{i+j+k=p}\tau(i)\tau(j)\tau(k) \equiv \begin{cases}0 & \text{if } (-71/p)=-1\\ 0 & \text{if } (-71/p)=0\\ 0 & \text{if } (-71/p)=1 \text{ and } p=x^2+xy+18y^2\\ 0 & \text{if } (-71/p)=1 \text{ and } p=2x^2\pm xy+9y^2\\ 1 & \text{if } (-71/p)=1 \text{ and } p=4x^2\pm 3xy+5y^2\\ -1 & \text{if } (-71/p)=1 \text{ and } p=3x^2\pm xy+6y^2\end{cases} \pmod{71}$$

Conjecture 4. *It appears that we also have the refinement that for $p\neq 71$:*

$$p = x^2+71y^2 \implies \sum\nolimits_{i+j+k=p}\tau(i)\tau(j)\tau(k)\equiv 0 \pmod{71^2}\,.$$

Though again, the form of the congruence modulo 71^3 is not clear.

5. Quadratic forms of discriminant -95

We can play the same game as above to obtain some results for discriminant -95, where $h(-95) = 8$. Unfortunately 95 is not prime, so we do not obtain directly any congruence for the 4-fold convolution $\tau^{*4}(n)$.

The class group is $\mathcal{C}(-95) \cong \mathbb{Z}/8\mathbb{Z}$, We have the following 8 quadratic forms

$$\underbrace{x^2+xy+24y^2}_{e(x,y)}, \quad \underbrace{2x^2+xy+12y^2}_{Q(x,y)}, \quad \underbrace{4x^2+xy+6y^2}_{Q^2(x,y)}, \quad \underbrace{3x^2-xy+8y^2}_{Q^3(x,y)},$$
$$\underbrace{5x^2+5xy+6y^2}_{Q^4(x,y)}, \quad \underbrace{3x^2+xy+8y^2}_{Q^5(x,y)}, \quad \underbrace{4x^2-xy+6y^2}_{Q^6(x,y)}, \quad \underbrace{2x^2-xy+12y^2}_{Q^7(x,y)}.$$

They split into the following genera, with the indicated congruence conditions

$$\begin{aligned}
&\mathcal{G}_1 = \{Q_0 = x^2+xy+24y^2, Q_{2,6} = 4x^2 \pm xy + 6y^2, Q_4 = 5x^2+5xy+6y^2\} \\
&\quad \overset{\text{represents}}{\leftrightarrow} \{1,4,6,9,11,16,24,26,36,39,44,49,54,61,64,66,74,81\} \subset (\mathbb{Z}/95\mathbb{Z})^* \\
&\mathcal{G}_2 = \{Q_1 = 2x^2 \pm xy + 12y^2, 3x^2 \pm xy + 8y^2\} \\
&\quad \overset{\text{represents}}{\leftrightarrow} \{2,3,8,12,13,18,22,27,32,33,37,48,52,53,67,72,78,88\} \subset (\mathbb{Z}/95\mathbb{Z})^*
\end{aligned}$$

These 8 quadratic forms give rise to 5 distinct theta series, namely

$$\begin{aligned}
\Theta_0 &= 1 + 2q + 2q^4 + 2q^9 + \cdots \\
\Theta_1 &= 1 + 2q^2 + 2q^8 + 2q^{12} + \cdots \\
\Theta_2 &= 1 + 2q^4 + 2q^6 + 2q^9 + 2q^{11} + \cdots \\
\Theta_3 &= 1 + 2q^3 + 2q^8 + 2q^{10} + 4q^{12} + \cdots \\
\Theta_4 &= 1 + 2q^5 + 4q^6 + \cdots,
\end{aligned}$$

modular for $\Gamma_0(95)$ with character ε_{-95}. It turns out that we have

$$\begin{aligned}
\frac{1}{2}(\Theta_0 - \Theta_2) &= \eta(5z)\eta(19z) \\
\frac{1}{2}(\Theta_2 - \Theta_4) &= \eta(z)\eta(95z)
\end{aligned}$$

We can therefore distinguish all forms in the principal genus. Unfortunately, since none of these η-products involve Θ-series of quadratic forms from the other genera, we cannot distinguish these forms.

Moreover, if we consider these modular forms, modulo 5, 19 or 95, we do not obtain any particular pleasant congruences for Ramanujan-τ. All we obtain is that

$$\begin{aligned}
\eta(5z)\eta(19z) &\equiv \eta(z)^5\eta(19z) \pmod 5 \\
\eta(5z)\eta(19z) &\equiv \eta(5z)\eta(z)^{19} \pmod{19} \\
\eta(z)\eta(95z) &\equiv \eta(z)\eta(19z)^5 \mod 5 \\
\eta(z)\eta(95z) &\equiv \eta(z)\eta(5z)^{19} \mod 19
\end{aligned}$$

To obtain an interesting congruence, we would want to say that $(1-z^{95n}) \equiv (1-z^n)^{95}$, modulo something. But unfortunately the binomial coefficients involved are coprime, so no such congruence follows from this method.

Question 6. Are there any interesting congruences on the convolution $\tau^{*4}(n)$, modulo 95, or some other modulus? What techniques can be used to find/prove them given the above fails? Or can the above congruences be used to obtain identities/results for other functions in this case?

Question 7. It appears that the η-products correspond to linear combinations of Θ-series from the principal genus. Is this always the case? How to characterise the primes represented by the forms in the non-principal genera? Can it be done in a simple way with 'familiar' modular forms, or does one have to resort to rational functions in η to obtain explicit identities?

6. Quadratic forms of discriminant $1 - 24N$

It seems already well-known that the class number of $h(1-24N)$ is 'relatively' large (in vague terms). We have $h(-23) = 3, h(-47) = 5, h(-71) = 7, h(-95) = 8, h(-119) = h(-143) = 10, h(-167) = 11$, and so on. (This is not increasing, as $h(-263) = 13$ is less than the previous term $h(-239) = 15$.)

We should focus on discriminant $D = 1-24N$, such that $|D|$ is *prime*. (This still does not make $h(1-24n)$ increasing, as the previous counterexample still holds.) In these cases we can obtain a congruence on the convolution $\tau^{*N}(n)$ modulo $24N-1$, at least if we can relate $\eta(z)\eta(Nz)$ to the theta series of quadratic forms of discriminant $D = 1-24N$.

Conjecture 5. *If $N > 1$ is such that $24N-1$ is prime, we have*

$$\eta(z)\eta(Nz) = \frac{1}{2}(\Theta_i - \Theta_j)$$

for some Θ-series $\Theta_i = \Theta_{Q_i}$ and $\Theta_j = \Theta_{Q_j}$. Thus we obtain the congruence

$$\sum_{a_1+\cdots+a_N} \tau(a_1)\cdots\tau(a_N) \equiv \begin{cases} 1 & \text{if } p = Q_i(x,y) \\ -1 & \text{if } p = Q_j(,y) \\ 0 & \text{otherwise} \end{cases} \pmod{24N-1}.$$

Moreover, we have the refinement that for $p \neq 24N-1$:

$$p = x^2 + (24N-1)y^2 \implies \sum_{a_1+\cdots+a_N=p} \tau(a_1)\cdots\tau(a_N) \equiv 0 \pmod{(24N-1)^2}.$$

Question 8. Can $\eta(z)\eta(Nz)$ always be written as a combination of Θ-series? It is always a combination of 2 Θ-series, and if so which ones?

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