Self-Similar Sums of Squares*

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Observation: $12^2 + 33^2 = 1233$. How many other examples can you find?

We're trying to find solutions to the equation $a^2 + b^2 = a \oplus b$, where $\oplus$ means concatenate their decimal expansion. If b has k digits, then this can be written:

$$a^2 + b^2 = 10^k a + b$$

Write this as:

$$a^2 - 10^k a + b^2 - b = 0$$

and solve this as a quadratic equation for a. We get:

$$a = \frac{10^k \pm \sqrt{10^{2k} - 4(b^2 - b)}}{2}$$

Firstly, this tells us that for every b, there are two possible a: a and $10^k - a$, when b has k digits. With the example above $k = 2$, so we get the other possible a as $100 - 12 = 88$. So $88^2 + 33^2 = 8833$.

The quadratic equation has an integer solution if and only if $10^{2k} - 4(b^2 - b)$ is a perfect square. So we reduce this to solving:

$$10^{2k} - 4(b^2 - b) = N^2$$

or after some rewriting:

$$10^{2k} + 1 = N^2 + (2b - 1)^2$$

At this point we're in the realm of number theory – solving a quadratic Diophantine equation. There is a fairly standard method for finding all solutions to this equation (for fixed k). Move to $\mathbb{Z}[\mathrm{i}]$, then the right hand side factorises as the norm of the Gaussian integer $N + (2b - 1)\mathrm{i}$. Since $\mathbb{Z}[\mathrm{i}]$ is a UDF, we can find all solutions by finding the prime decomposition of $10^{2k} + 1$ in $\mathbb{Z}[\mathrm{i}]$, and equating the decompositions of both sides. To decompose $10^{2k} + 1$ in $\mathbb{Z}[\mathrm{i}]$, factor it in $\mathbb{Z}$, and write each prime factor as the sum of two squares if possible.

For example, take $k = 5$:

$$\begin{aligned} 10^{2\times 5} + 1 &= 101 \times 3541 \times 27961 \\ &= (10^2 + 1^2) \times (54^2 + 25^2) \times (144^2 + 85^2) \\ &= (10 + 1\mathrm{i})(10 - 1\mathrm{i}) \times (54 + 25\mathrm{i})(54 - 25\mathrm{i}) \times (144 + 85\mathrm{i})(144 - 85\mathrm{i}) \end{aligned}$$

Uniqueness of prime factorisation in $\mathbb{Z}[\mathrm{i}]$ means:

$$N + (2b - 1)\mathrm{i} = \mathrm{i}^\ell (10 + 1\mathrm{i})^a (10 - 1\mathrm{i})^b \times (54 + 25\mathrm{i})^c (54 - 25\mathrm{i})^d \times (144 + 85\mathrm{i})^e (144 - 85\mathrm{i})^f$$

*Name taken from `http://web.science.mq.edu.au/~alf/SomeRecentPapers/174a.pdf`, a paper dealing with the same question, and giving a different style of solution.

where $\ell = 0, 1, 2,$ or 3, and $a + b = 1$, $c + d = 1$, $e + f = 1$.

With $a = 1, c = 1, e = 1$, and $\ell = 0$, we get:

$$N + (2b - 1)\mathrm{i} = 48320 + 87551\mathrm{i}$$

We read off $N = 48320$, and $2b - 1 = 87551 \implies b = 43776$ which indeed has 5 digits. Plug back in to get $a = 25840$.

So we get another example:

$$25840^2 + 43776^2 = 2584043776$$

Modulo things like decomposing 10^{2k+1} as a product of primes, and writing these as the sum of two squares, at this point it is reasonably straight-forward to program a computer to generate all possible solutions.

Doing so, one will eventually stumble on something like:

$$88321167\,88321167\,883212^2 + 32116788\,32116788\,321168^2 = \\ 8832116788321167883212321167883211678832116 8$$

and wonder whether adding any number of the blocks 88321167 and 32116788 gives a solution.

To prove this is the case, write out an expression for such a and b, and sum the geometric series in each:

$$\begin{aligned}
a &= \sum_{i=0}^{n} 88321167 \times 10^{8i+6} + 883212 \\
&= 88321167 \times 10^6 \times \left(\frac{10^{8n+8} - 1}{10^8 - 1}\right) + 883212 \\
&= \frac{121}{137} \times 10^{8n+14} + \frac{44}{137} \\
b &= \sum_{i=0}^{n} 32116788 \times 10^{8i+6} + 321168 \\
&= 32116788 \times 10^6 \times \left(\frac{10^{8n+8} - 1}{10^8 - 1}\right) + 321168 \\
&= \frac{44}{137} \times 10^{8n+14} + \frac{16}{137}
\end{aligned}$$

In this case b has $8(n + 1) + 6 = 8n + 14$ digits. So we confirm this gives an infinite family of solutions by checking whether:

$$a^2 + b^2 = 10^{8k+14}a + b$$

This is just a simple case of multiplying out:

$$\begin{aligned}
a^2 + b^2 &= \left(\frac{121}{137} \times 10^{8n+14} + \frac{44}{137}\right)^2 + \left(\frac{44}{137} \times 10^{8n+14} + \frac{16}{137}\right)^2 \\
&= \frac{121^2 + 44^2}{137^2} \times 10^{16n+28} + \frac{2 \times 121 \times 44 + 2 \times 44 \times 16}{137^2} 10^{8n+14} + \frac{44^2 + 16^2}{137^2} \\
&= \frac{121}{137} \times 10^{16n+28} + \frac{88}{137} \times 10^{8n+14} + \frac{16}{137}
\end{aligned}$$

verses:

$$
\begin{aligned}
10^{8n+14}a + b &= 10^{8n+14}\left(\frac{121}{137} \times 10^{8n+14} + \frac{44}{137}\right) + \left(\frac{44}{137} \times 10^{8n+14} + \frac{16}{137}\right) \\
&= \frac{121}{137} \times 10^{16n+28} + \frac{2 \times 44}{137} 10^{8n+14} + \frac{16}{137} \\
&= \frac{121}{137} \times 10^{16n+28} + \frac{88}{137} \times 10^{8n+14} + \frac{16}{137}
\end{aligned}
$$

And they agree!

Observing some of the simplifications that happen in this proof, one can reverse engineer it as a method to producing candidate families for some other solutions. For example we find that prepending:

97490513219843586716000250948678015641328399 to 97490513220 and

15641328399974905132198435867160002509486780 to 15641328400

gives another infinite family of solutions.