First Year Progression Report

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1 Introduction

The polylogarithms $\mathrm{Li}_s(z)$ are an important and frequently occurring class of functions, with applications throughout mathematics and physics. In this report I will give an overview of some of the theory surrounding them, and the questions and lines of research still open to investigation.

I will introduce the polylogarithms as a generalisation of the logarithm function, by means of their Taylor series expansion, multiple polylogarithms will following by considering products. This will lead to the idea of functional equations for polylogarithms, capturing the symmetries of the function. I will give examples of such functional equations for the dilogarithm. One of the main problems in the theory concerns finding non-trivial functional equations for the higher polylogarithms.

Then I will explore the connection between polylogarithms and the Dedekind zeta function of a number field. The Dedekind zeta function $\zeta_F(z)$ is an important invariant of a number field, capturing much arithmetic data, including the class number h_F, discriminant Δ_F and regulator Reg_F, in its behaviour at the point $z = 1 \in \mathbb{C}$. Its value at other special points should also provide significant arithmetic information the number field. Zagier conjectures, in a precise way, that the value of $\zeta_F(z)$ at a positive integer n is given in terms of order n polylogarithms.

I will move on to the notion of iterated integrals $I(a_0; a_1, \ldots, a_n; a_{n+1})$, and a purely algebraic lifting of them in the form of Goncharov's motivic iterated integrals $I^{\mathcal{M}}(a_0; a_1, \ldots, a_n; a_{n+1})$. Lifting to motivic iterated integrals introduces new algebraic structures not visible on the level of numbers, particularly the Hopf algebera coproduct Δ. Iterated integrals can be used to encode polylogarithms, so motivic viewpoint should imbue additional structure on them.

Then I will come to the polygon algebra of Gangl, Goncharov and Levin. It provides a way of capturing important combinatorial properties of multiple polylogarithms, in their iterated integral form, by means of polygons decorated form a set R. The differential on the polygon algebra mimics a differential on algebraic cycles, which themselves correspond to multiple polylogarithms. Polygons admit many other internal structures. Agarwala introduces more general differentials on polygons which helps in exploring the dihedral symmetries of multiple logarithms. Polygons have an operadic structure, a way of gluing them together, which bears similarities to the mosaic operad that appears in connection with tessellations of moduli spaces. The polygon algebra also admits a differential by collapsing vertex-vertex arrows, a similar structure occurs in the coproduct of Dupont's dissection polylogarithms.

Next I will turn to the notion of a multiple zeta value (abbreviated MZV) $\zeta(n_1, \ldots, n_k)$, a generalisation of a Riemann zeta value $\zeta(n)$ to multiple arguments, which is defined by an infinite series and is nothing more than a special value of the multiple polylogarithm. The transcendence properties of MZVs are very mysterious, with numerous conjectures having entered mathematical folklore, but little provably known. One expects all relations between MZVs to be homogeneous in weight, one expects certain MZVs to be algebraically independent, one has an explicit conjectural formula for the dimension of weight k MZVs and suggestions for a basis. A standard family of relations between MZVs can be described by comparing two distinct multiplicative structures, and these double shuffle relations are expected to generate all relations between MZVs.

Lastly I will introduce Francis Brown's notion of a motivic MZV $\zeta^{\mathfrak{m}}(n_1, \ldots, n_k)$, which further lifts Goncharov's motivic iterated integrals. These motivic MZVs have recently been used to provide partial proofs for some of the MZV folklore conjectures: we get spanning sets and bounds on the dimension of the space of MZVs of weight k. In this lifting we gain a new algebraic structure, a coaction, which forms the basis of a decomposition algorithm on motivic MZVs, and provides combinatorial tools to study MZV relations. I will use this coaction to investigate some known and conjectured relations between MZVs.

2 Polylogarithms

2.1 Definitions

As a generalisation of the of the ordinary logarithm function, the order s polylogarithm is defined to be:

$$\mathrm{Li}_s(z) \coloneqq \sum_{n=1}^{\infty} \frac{1}{n^s} z^n$$

which converges inside the unit disc $|z| < 1$. This function was first considered by Leibniz and Bernoulli in 1696, [32].

Taking $s = 1$, one finds:

$$\mathrm{Li}_1(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = -\log(1-z)$$

so this does genuinely generalise the logarithm function.

Computing the derivative of $\mathrm{Li}_s(z)$, one finds that:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Li}_s(z) = \frac{1}{z} \mathrm{Li}_{s-1}(z)$$

which means the polylogarithm may be analytically continued to the cut complex plane $\mathbb{C} \setminus [1, \infty)$ by:

$$\mathrm{Li}_s(z) = \int_0^z \mathrm{Li}_{s-1}(t) \frac{\mathrm{d}t}{t}$$

More generally, one can also define the *multiple* polylogarithm as in [26]:

$$\mathrm{Li}_{s_1, s_2, \ldots, s_m}(z_1, z_2, \ldots, z_m) \coloneqq \sum_{0 < n_1 < n_2 < \cdots < n_m} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_m^{s_m}} z_1^{n_1} z_2^{n_2} \cdots z_m^{n_m}$$

Multiple polylogarithms would arise naturally enough when looking at the product of two polylogarithms. Consider:

$$\mathrm{Li}_s(z_1) \, \mathrm{Li}_t(z_2) = \sum_{n=1}^{\infty} \frac{1}{n^s} z_1^n \sum_{m=1}^{\infty} \frac{1}{m^t} z_2^m = \sum_{n,m>0} \frac{1}{n^s m^t} z_1^n z_2^m$$

The double sum breaks up into pieces where $n < m$, $n = m$ or $n > m$, giving:

$$\begin{aligned} &= \sum_{0<n<m} \frac{1}{n^s m^t} z_1^n z_2^m + \sum_{0<n=m} \frac{1}{n^s m^t} z_1^n z_2^m + \sum_{n>m>0} \frac{1}{n^s m^t} z_1^n z_2^m \\ &= \mathrm{Li}_{s,t}(z_1, z_2) + \mathrm{Li}_{s+t}(z_1 z_2) + \mathrm{Li}_{t,s}(z_2, z_1) \end{aligned}$$

and each term can be recognised as a polylogarithm, or multiple polylogarithm.

Polylogarithms and multiple polylogarithms attract considerable interest because appear in a vast number of disparate areas of mathematics and physics. On the physics side, they appear as closed form solutions for the Fermi-Dirac and Bose-Einstein distributions [36], and in the computation of Feynman diagram integrals [39].

In mathematics they occur in algebraic K-theory [3] and in the computation of the volume of hyperbolic manifolds [25, 35]. In connection with the value of L-functions, they form an integral part of *Zagier's polylogarithm conjecture* which ultimately concerns special values of the Dedekind zeta function [41]. Certain special values of multiple polylogarithms gives us the multiple zeta values, a class of numbers which attract significant interest in their own right.

2.2 Special Values, Functional Equations and Singled Valued Versions

The dilogarithm is the only mathematical function with a sense of humour. – Zagier

The dilogarithm is the order $s = 2$ polylogarithm, the first polylogarithm beyond the elementary log function. According to Zagier [44], in contrast to most other functions which have either no exactly computable special values, or have a countable easily describable set, the dilog seems to have only a scattered few:

$$\begin{gathered}
\operatorname{Li}_2(0)=0, \quad \operatorname{Li}_2(1)=\tfrac{\pi^2}{6}, \quad \operatorname{Li}_2(-1)=-\tfrac{\pi^2}{12}, \quad \operatorname{Li}^2\left(\tfrac{1}{2}\right)=\tfrac{\pi^2}{12}-\tfrac{1}{2}\log^2(2), \\
\operatorname{Li}_2\left(\tfrac{3-\sqrt{5}}{2}\right)=\tfrac{\pi^2}{15}-\log^2\left(\tfrac{1+\sqrt{5}}{2}\right), \quad \operatorname{Li}_2\left(\tfrac{-1+\sqrt{5}}{2}\right)=\tfrac{\pi^2}{10}-\log^2\left(\tfrac{1+\sqrt{5}}{2}\right), \\
\operatorname{Li}_2\left(\tfrac{1-\sqrt{5}}{2}\right)=-\tfrac{\pi^2}{15}+\tfrac{1}{2}\log^2\left(\tfrac{1+\sqrt{5}}{2}\right), \quad \operatorname{Li}_2\left(\tfrac{-1-\sqrt{5}}{2}\right)=-\tfrac{\pi^2}{10}+\tfrac{1}{2}\log^2\left(\tfrac{1+\sqrt{5}}{2}\right)
\end{gathered}$$

But compared with its limited number of special values, the dilogarithm satisfies a huge number of functional equations. ‘Trivially’ there is a inversion formula:

$$\operatorname{Li}_2\left(\frac{1}{z}\right)=-\operatorname{Li}_2(z)-\frac{\pi^2}{6}-\frac{1}{2}\log^2(-z)$$

a duplication formula:

$$\operatorname{Li}_2(z^2)=2(\operatorname{Li}_2(z)+\operatorname{Li}_2(-z))$$

Less trivially a reflection formula:

$$\operatorname{Li}_2(1-z)=-\operatorname{Li}_2(z)+\frac{\pi^2}{6}-\log(z)\log(1-z)$$

Foremost among the functional equations is the two-variable *five-term* relation:

$$\begin{gathered}
\operatorname{Li}_2(x)+\operatorname{Li}_2(y)+\operatorname{Li}_2\left(\frac{1-x}{1-xy}\right)+\operatorname{Li}_2(1-xy)+\operatorname{Li}_2\left(\frac{1-y}{1-xy}\right)= \\
\frac{\pi^2}{6}-\log(x)\log(1-x)-\log(y)\log(1-y)+\log\left(\frac{1-x}{1-xy}\right)\log\left(\frac{1-y}{1-xy}\right)
\end{gathered}$$

Here the entire right hand side is a combination of elementary functions.

Higher polylogarithms are known to satisfy various trivial functional equations for all orders. At least for small orders they also satisfy non-trivial functional equations, with such functional equations expected for all orders [20].

At this point is is also worth introducing a variant of the dilogarithm. The Bloch-Wigner dilogarithm is defined by:

$$D(z):=\operatorname{Im}(\operatorname{Li}_2(z))+\arg(1-z)\log|z|$$

This defines us a single-valued, real analytic continuous function on $\mathbb{C}\setminus\{0,1\}$, by eliminating the jump of $2\pi\mathrm{i}\log|z|$ when crossing the branch cut $(1,\infty)$.

The Bloch-Wigner dilog satisfies functional equations of its own corresponding to those of the dilogarithm. The Bloch-Wigner functional equations are simpler because they do not contain the lower-order logarithms terms. For example the five-term relations becomes exactly:

$$D(x)+D(y)+D\left(\frac{1-x}{1-xy}\right)+D(1-xy)+D\left(\frac{1-y}{1-xy}\right)=0$$

A similar variant of the general polylogarithm $\operatorname{Li}_m(z)$ can be given by:

$$P_m(z)=\left.\begin{matrix}\operatorname{Re}(\cdot)\text{ if } m \text{ odd}\\ \operatorname{Im}(\cdot)\text{ if } m \text{ even}\end{matrix}\right\}\sum_{k=0}^{m-1}\frac{2^k B_k}{k!}\log^k|z|\operatorname{Li}_{m-k}(z)$$

where B_k is the k-th Bernoulli number [41]. It will be necessary to use this function when making the link to special values of the Dedekind zeta function.

2.3 The Dedekind Zeta Function and Polylogarithms

One significant area where polylogarithms appear is in connection with the Dedekind zeta function of a number field, or more specifically its value at special points.

As a generalisation of the Riemann zeta function, the Dedekind zeta function associated to the number field F is defined by:

$$\zeta_F(s) \coloneqq \sum_{I \neq (0) \subset \mathcal{O}_F} \frac{1}{\mathrm{N}(I)^s}$$

where the sum is taken over all non-zero, integral ideals of F , and $\mathrm{N}(I)$ is the ideal norm of I. The existence and uniqueness of factorisation of an ideal into prime ideals in a number field is reflected analytically by the fact that the Dedekind zeta function admits an Euler product expansion:

$$\zeta_F(s) = \prod_{\mathfrak{p} \neq (0) \subset \mathcal{O}_F} \frac{1}{1 - \mathrm{N}(\mathfrak{p})^{-s}}$$

where the product runs over all non-zero prime ideals of $\mathcal{O}_F$, [Section 10.5.1 in 12].

The series above for $\zeta_F(s)$ converges for $\mathrm{Re}(s) > 1$, but $\zeta_F(s)$ can be analytically continued to a meromorphic function on the complex plane $\mathbb{C}$, with only one simple pole at $s = 1$.

When looking at the number field $F = \mathbb{Q}$, the Dedekind zeta function $\zeta_{\mathbb{Q}}(s)$ is simply the usual Riemann zeta function $\zeta(s)$. In this case, the Riemann zeta function gives significant arithmetic information about $\mathbb{Q}$ and its ring of integers $\mathbb{Z}$, particularly about the distribution of prime numbers. So one would naturally expect the Dedekind zeta function $\zeta_F(s)$ to give similar information about the number field F, its ring of integers $\mathcal{O}_F$, and the distribution of prime ideals in $\mathcal{O}_F$.

An instance of this is in the so-called *analytic class number formula* for the residue of $\zeta_F(s)$ at $s = 1$, which is given in terms of some arithmetic data of the number field F.

Theorem 2.1 (Analytic Class Number Formula, [Theorem 10.5.1 in 12])**.** *Give a number field F, the residue of the Dedekind zeta function $\zeta_F(s)$ at its simple pole $s = 1$ is given by:*

$$\mathrm{Res}_{s=1}\, \zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} h_F \,\mathrm{Reg}_F}{w_F \sqrt{|\Delta_F|}}$$

where:

- r_1 *is the number of real embeddings of* F*,*
- r_2 *is the number of conjugate pairs of complex embeddings of* F*,*
- h_F *is the class number of* F*,*
- w_F *is the number of roots of unity* F *contains,*
- Δ_F *is the discriminant of* F*, and*
- Reg_F *is the regulator of* F*.*

The regulator Reg_F of the number field F is defined by taking the volume of a fundamental domain for the lattice spanned by the units of F in logarithmic space. This leads to an expression for Reg_F in terms of the logarithm of elements of F. This can be seen as relating the 'value' of $\zeta_F(1)$, or rather the 'value' of $\zeta_F(1)/\zeta(1)$, to the first polylogarithm $-\log(1-x)$. This is a first instance of Zagier's conjecture.

The next instance comes from considering $\zeta_F(2)$, and taking a detour through hyperbolic space. The volume of an ideal tetrahedron in hyperbolic 3-space, with vertices $\{0, 1, \infty, z\}$ is given by $D(z)$, with D the Bloch-Wigner dilogarithm defined earlier, [35]. For F an imaginary quadratic field, the group $\mathrm{SL}_2(\mathcal{O}_F)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$, therefore acts on hyperbolic space $\mathbb{H}^3$ by isometries.

Humbert's volume formula [27] gives the volume of the quotient space as:

$$\mathrm{Vol}(\mathbb{H}^3/\,\mathrm{SL}_2(\mathcal{O}_F)) = \frac{|\Delta_k|^{3/2}}{4\pi^2}\zeta_F(2)$$

But this quotient space can be triangulated by ideal tetrahedron with vertices in $\mathbb{P}^1(F)$ inside the boundary $\mathbb{P}^1(\mathbb{C})$ of $\mathbb{H}^3$. This means:

$$\zeta_F(2) = \frac{\pi^2}{3\,|\Delta_F|^{3/2}} \sum_\nu n_\nu D(z_\nu)$$

for some $z_\nu \in F$. So $\zeta_F(2)$ can be expressed in terms of dilogarithms of elements of the number field F. Zagier [p. 17 of 41] gives the following example for $F = \mathbb{Q}(\sqrt{-7})$:

$$\zeta_F(2) = \frac{4\pi^2}{21\sqrt{7}} \left(2D\left(\frac{1+\sqrt{-7}}{2}\right) + D\left(\frac{-1+\sqrt{-7}}{4}\right)\right)$$

A similar analysis works for general number fields [40], and leads to the result that:

$$\frac{\sqrt{|\Delta_F|}}{\pi^{2(r_1+r_2)}}\zeta_F(2)$$

is a rational multiple of an $r_2 \times r_2$ determinant with entries $\sum n_i D(z_i)$, $x_i \in F$.

By finding elements in the *Bloch group* of F, we can use the theorem in Part I, Section 1 of [45], to evaluate $\zeta_F(2)$ in other cases, up to some rational factor. I find elements in the Bloch group with computer assistance.

Example 2.2. Consider the number field $F = \mathbb{Q}(\sqrt{-5})$, which has $r_1 = 0$, $r_2 = 1$, discriminant -20. I find:

$$\zeta_F(2) = \frac{\pi^2}{30\sqrt{20}} \left(4D(2+\sqrt{-5}) + 3D(\tfrac{2+\sqrt{-5}}{4}) + 20D(\tfrac{2+\sqrt{-5}}{3})\right)$$

which is consistent with the result above.

This result corresponds to the fact that:

$$4[2+\sqrt{-5}] + 3[\tfrac{2+\sqrt{-5}}{4}] + 20[\tfrac{2+\sqrt{-5}}{3}]$$

is an element of the Bloch group $\mathcal{B}(F)$.

Now consider $F = \mathbb{Q}(\alpha)$, where α is a root of $x^3 + x - 1$. This field has $r_1 = 1$, $r_2 = 1$, and discriminant -31. Then:

$$\zeta_F(2) = \frac{4\pi^4}{93\sqrt{31}}\left(2D\sigma(-1-2\alpha^2) + 2D\sigma(2-\alpha) - 2D\sigma(1+2\alpha^2)\right)$$

where σ is the embedding $F \hookrightarrow \mathbb{C}$, which sends α to the root $-0.34\ldots + 1.16\ldots\mathrm{i}$ of $x^3 + x - 1$ with strictly positive imaginary part.

This corresponds to the fact that:

$$2[-1-2\alpha^2] + 2[2-\alpha] - 2[1+2\alpha^2]$$

is an element of the Bloch group $\mathcal{B}(F)$.

Results like this are expected to holds when evaluating $\zeta_F(n)$ for any integer n, connected with higher analogues of the Bloch group, and the higher polylogarithms. As a simple example of this, for $F = \mathbb{Q}(\sqrt{-5})$, as above, we have:

$$\zeta_F(3) = \frac{3\pi^3}{35\sqrt{20}} 2P_3(\tfrac{1}{2})$$

The general case is given in its abstract algebraic K-theory formulation in [41] by:

Conjecture 2.3 (Zagier's Polylogarithm Conjecture)**.** *There is a quasi-isomorphism ψ making the following diagram commute:*

$$\begin{array}{ccc} K_{2m-1}(F) & \xrightarrow{\quad\psi\quad} & \mathcal{B}_m(F) \\ & {\scriptstyle \mathrm{Reg}_F^m}\searrow \quad \swarrow{\scriptstyle (P_m,\ldots,P_m)} & \\ & \mathbb{R}^{r\pm} & \end{array}$$

where $K_{2m-1}(F)$ is an algebraic K-group of F, and $B_m(F)$ is a Bloch group.

As a consequence of this, the formulation in terms of special values of the Dedekind zeta function, taken from Conjecture 1 in [13], reads as follows:

Conjecture 2.4. *If F is a number field, and $\sigma_{1+r_2}, \ldots, \sigma_{r_1+r_2} \colon F \to \mathbb{R}$ are its real embeddings, and $\sigma_1 = \overline{\sigma_{1+r_1+r_2}}, \ldots, \sigma_{r_2} = \overline{\sigma_{r_1+2r_2}} \colon F \to \mathbb{C}$ are its complex embeddings. Let $n \geq 2$ be an integer, write $d_n = r_1 + r_2$ if n odd, and $d_n = r_2$ if n even. Then there exists elements $y_1, \ldots, y_{d_n} \in \mathbb{Q}[F \setminus \{0,1\}]$ such that, up to $\mathbb{Q}^*$:*

$$\zeta_f(n) = \pi^{(r_1+2r_2-d_n)n} \left|\Delta_F\right|^{-1/2} \det(P_n(\sigma_i(y_j)))$$

where $1 \leq i, j \leq d_n$.

So far Zagier's conjecture has only been proven for $n \leq 3$, it follows from the work of Bloch and Suslin in the case $n = 2$, and by Goncharov [24] in the case $n = 3$.

2.4 Iterated Integrals and Their Properties

For complex numbers x_i, an iterated integral, introduced by Chen [11], is defined by:

$$I(x_0; x_1, \ldots, x_m; x_{m+1}) \coloneqq \int_{\Delta_\gamma} \frac{\mathrm{d}t_1}{t_1 - x_1} \wedge \cdots \wedge \frac{\mathrm{d}t_m}{t_m - x_m}$$

where γ is a path from x_0 to x_{m+1} in $\mathbb{C} \setminus \{x_1, \ldots, x_m\}$, and the region of integration Δ_γ consists of all m-tuples $(\gamma(t_1), \ldots, \gamma(t_m))$, with $t_1 \leq t_2 \leq \cdots \leq t_m$.

It is possible to write the multiple polylogarithms defined above as iterated integrals. One has:

$$\mathrm{Li}_{n_1,\ldots,n_k}(z_1, \ldots, z_k) = (-1)^k I_{n_1,\ldots,n_k}\left(\frac{1}{z_1 \cdots z_k}, \ldots, \frac{1}{z_k}\right)$$

where we have used the shorthand:

$$I_{n_1,n_2,\ldots,n_k}(x_1, x_2, \ldots, x_k) \coloneqq I(0; \underbrace{x_1, 0, \ldots, 0}_{n_1 \text{ arguments}}, \underbrace{x_2, 0, \ldots, 0}_{n_2 \text{ arguments}}, \ldots, \underbrace{x_k, 0, \ldots, 0}_{n_k \text{ arguments}}; 1)$$

The iterated integrals $I(x_0; x_1, \ldots, x_n; x_{n+1})$ satisfy a number of standard and well-known properties, listed below:

– Equal boundaries: $I(x_0; x_1, \ldots, x_n; x_{n+1}) = 0$ if $x_0 = x_{n+1}$, and $n \geq 1$.
– Empty integral/unit: $I(x_0; x_1) = 1$, for any x_0, x_1.
– Path composition: for fixed y:

$$I(x_0; x_1, \ldots, x_n; x_{n+1}) = \sum_{k=0}^{m} I(x_0; x_1, \ldots, x_k; y) I(y; x_{k+1}, \ldots, x_m; x_{m+1})$$

The product of two iterated integrals is given by the shuffle product of their parameters:

$$\begin{aligned} I(a; x_1, \ldots, x_m; b) I(a; x_{m+1}, \ldots, x_{m+n}; b) &= I(a; \{x_1, \ldots, x_m\} \sqcup \{x_{m+1}, \ldots, x_{m+n}\}; b) \\ &= \sum_{\sigma \in S_{m,n}} I(a; x_{\sigma(1)}, \ldots, x_{\sigma(m)}) \end{aligned}$$

Here $S_{m,n}$ is the set of (n,m)-shuffles, those permutations σ in S_{m+n} satisfying $\sigma(1) < \sigma(2) < \cdots < \sigma(m)$, and $\sigma(m+1) < \sigma(m+2) < \cdots < \sigma(m+n)$. This means the set $\{1, 2, \ldots, m+n\}$ is permuted, but the subsets $\{1, 2, \ldots, m\}$ and $\{m+1, m+2, \ldots, m+n\}$ maintain the same ordering – they are merely shuffled.

Together these properties imply reversal of paths:

$$I(x_0; x_1, \ldots, x_n; x_{n+1}) = (-1)^n I(x_{n+1}; x_n, \ldots, x_1, x_0)$$

2.5 The Hopf algebra of (Motivic) Iterated Integrals

Goncharov [22] shows how these iterated integrals $I(x_0; x_1, \ldots, x_n; x_{n+1})$, defined above, can be upgraded to *framed mixed Tate motives over* $\overline{\mathbb{Q}}$, at least when the parameters x_i are algebraic numbers. This gives a motivic iterated integral:

$$I^{\mathcal{M}}(x_0; x_1, \ldots, x_n; x_{n+1}) \in \mathcal{A}_n(\overline{\mathbb{Q}})$$

which by definition lies in a commutative, graded Hopf algebra $\mathcal{A}_\bullet(\overline{\mathbb{Q}})$.

Assuming the parameters x_i are algebraic numbers, since there are finitely many, one can suppose they lie in some number field F, rather than just in $\overline{\mathbb{Q}}$. Then the graded, commutative Hopf algebra $\mathcal{A}_\bullet(F)$ is the fundamental Hopf algebra of the abelian category $\mathcal{M}_T(F)$ of *mixed Tate motives over* F.

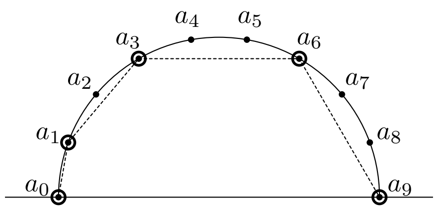
Since the motivic iterated integrals lie in a Hopf algebra, they admit a coproduct Δ. This is a genuinely new algebraic structure on iterated integrals; it is completely invisible at the level of numbers. Goncharov proves that the coproduct is given by:

$$\Delta I^{\mathcal{M}}(a_0; a_1, \ldots, a_n; a_{n+1}) =$$
$$\sum_{0 < i_0 < i_1 < \ldots < i_k < i_{k+1} = n+1} I^{\mathcal{M}}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \prod_{p=0}^{k} I^{\mathcal{M}}(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}})$$

This formula has an elegant interpretation in terms of cutting off segments of a semicircular polygon. For example, the term:

$$I^{\mathcal{M}}(a_0; a_1, a_3, a_6; a_9) \otimes I^{\mathcal{M}}(a_0; a_1) I^{\mathcal{M}}(a_1; a_2; a_3) I^{\mathcal{M}}(a_3; a_4, a_5; a_6) I^{\mathcal{M}}(a_6; a_7, a_8; a_9)$$

in the coproduct $\Delta I^{\mathcal{M}}(a_0; a_1, \ldots, a_8; a_9)$ corresponds to cutting off the indicated segments from the semicircular polygon below:



The other terms arise from taking all other possible choices of segments.

There is a cannonical surjective homomorphism:

$$p\colon I^{\mathcal{M}}(a_0; a_1, \ldots, a_n; a_{n+1}) \mapsto I(a_0; a_1, \ldots, a_n; a_{n+1})$$

which realises a motivic iterated integral by its classical counterpart, so that any relations satisfied on the motivic level also hold on the level of classical integrals.

Conjecturally, no information is lost when moving from classical to motivic iterated integrals, so that this in fact defines an isomorphism between motivic and classical integers; any relations between classical iterated integrals should lift to a motivic incarnation. Regardless, as a purely algebraic lifting, we gain the structure of a Hopf algebra, and eliminate the transcendence problems that plague classical iterated integrals.

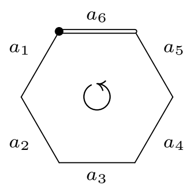
I will touch on these ideas more when I introduce Brown's motivic MZVs, which further lift a special class of iterated integrals.

2.6 The Polygon Algebra

In order to capture combinatorially, the properties of polylogarithms and multiple polylogarithms, Gangl, Goncharov, and Levin [18] define the tree and polygon algebras. Triangulated polygons are mapped to trees, and these trees map to *admissible algebraic cycles* using the *forest cycling map*. These algebraic cycles are an avatar, in a precise sense, of polylogarithms.

A R-deco polygon is an oriented polygon with distinguished root side, and whose edges are decorated with elements from a given set R. The orientation induces an ordering on the sides and vertices of the polygon; the root side is the last side, and the orientation is determined by marking the first vertex (the intersection of the first and root sides) with a bullet $\bullet$.

For example, this is an R-deco hexagon, with decorations from the set $R = \{a_1, a_2, a_3 \ldots\}$:



The polygon has anti-clockwise orientation (which is mathematically positive), so the side labelled a_1 is the first side, the side labelled a_2 is the second side, etcetera. Here, the orientation of the polygon is marked in the interior for clarity, but would normally be read off from the first vertex and root side. The first vertex is marked with the bullet $\bullet$, the root side distinguished by a double line – together this means first edge is the edge marked a_1, and the polygon is oriented anti-clockwise.

The weight $\chi(\pi)$ of an R-deco n-gon π is defined to be $n-1$, the number of non-root sides. So the R-deco hexagon above has weight 5.

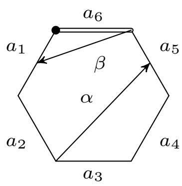
Assemble the R-deco polygons of weight n into a $\mathbb{Q}$-vector space $V_n(R)$, where we take $V_0(R) \coloneqq \mathbb{Q}$. The graded vector space of polygons is then defined by:

$$V_\bullet(R) \coloneqq \bigoplus_{n=0}^{\infty} V_n(R)$$

The polygon algebra is $P^*_\bullet = \bigwedge^* V_\bullet$, the exterior algebra of the grade vector space of polygons.

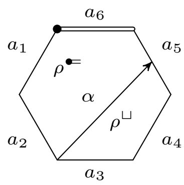
There is a differential on the polygon algebra, which mimics the differential on algbraic cycles, meaning the mappings from polygons to trees to algebraic cycles holds on the level of differential graded algebras. The differential is defined using arrows in the R-deco polygons.

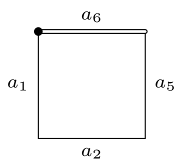
An arrow in an R-deco polygon π is a directed line segment beginning at a vertex of π, and ending at the interior of a side of π. For example, the arrows α and β below:

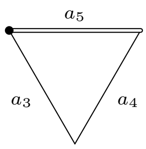


An arrow is backwards if in the linear ordering of sides and vertices, its end is before its start. For example the arrow β above, is backwards. The arrow α is forwards.

Given an arrow α, one associated a root polygon $\pi^{\bullet\!=}$ containing the root side and first vertex, and a cutoff polygon $\pi^{\sqcup}$ by collapsing the arrow α, and inducing a root orientation on the new polygons. The starting vertex determines the new first vertex, and the ending side the root side of the cutoff polygon:

$\pi =$ 

$\rightsquigarrow \pi^{\bullet\!=} =$ 

and $\pi^{\sqcup} =$ 

The differential $\partial\pi$ of an R-deco polygon π is defined by:

$$\partial\pi = \sum_{\text{arrows } \alpha} \operatorname{sgn}(\alpha)\pi^{\bullet\!=} \wedge \pi^{\sqcup}$$

where:

$$\operatorname{sgn}(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ forwards} \\ (-1)^{\chi(\pi^{\sqcup})} & \text{if } \alpha \text{ backwards} \end{cases}$$

This is genuinely a differential $\partial^2\pi = 0$. This differential is closely related to Goncharov's coproduct on iterated integrals defined in subsection 2.5. Adding more arrows to a polygon gives the notion of a dissection, which can be used to interpret the bar construction of the differential graded polygon algebra.

An extension of the polygon algebra, to allow for undecorated sides, gives us the promised correspondence with multiple polylogarithms. Roughly speaking, the polylogarithm $I_{n_1,\ldots,n_r}(a_1,\ldots,a_r)$ in iterated integral form corresponds to the R-deco polygon:

$$\pi = [\underbrace{a_1, \varnothing, \ldots, \varnothing}_{n_1 \text{ times}}, \ldots, \underbrace{a_r, \varnothing, \ldots, \varnothing}_{n_r \text{ times}}, 1]$$

Here the labels of the sides are given from first to root, with ∅ denoting an undecorated side. In [15] Duhr, Gangl, and Rhodes show how the *symbol* of a polylogarithm can be calculated from maximal dissections of the corresponding R-deco polygon. The symbol, or $\otimes^m$-invariant defined in [22], is an invariant of polylogarithms, living in a tensor algebra to make calculations easier, which reflects the combinatorial properties of polylogarithms and the functional equations they satisfy.

2.7 Algebraic Structures on Polygons

2.7.1 Other Differentials on Polygons

The obvious dihedral symmetries of polygons naturally raises the question of how multiple polylogarithm behave under dihedral symmetries. In [1], Agarwala defines a number of different differentials on the polygon algebra in an effort to study the dihedral symmetries of multiple polylogarithms.

She gives a general criterion for checking whether a particular 'rule' generates a differential by relating it to a Hopf algebra structure on the dual trees defined by dissections. This generalise the proof in [18] that ∂ is a differential on the polygon algebra.

By defining perturbations of these differentials, where the rule employed differs on certain carefully chosen sets of arrows, she is able to relate the rotations and reflections of polygons to the original polygon, modulo some co-ideal. This gives some corresponding relations on the level of multiple polylogarithms.

2.7.2 Operadic Structure of Polygons

It is reasonably clear that there is a natural way of gluing polygons together. More precisely, polygons have an operadic structure.

In order to motivate/explain this observation, I need to define what an operad is. Much more information about operads, beyond the following definition, is given in [33]. A (non-symmetric) operad $\mathcal{P}$ is a collection of k-vector spaces $\mathcal{P}(n)$ for $n = 0, 1, 2, \ldots$, whose elements should be though of as formal n-ary operations. There is a composition:

$$\gamma\colon \mathcal{P}(\ell) \otimes (\mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_\ell)) \to \mathcal{P}(n_1 + \cdots + n_\ell)$$

which amounts to plugging the output of ℓ operations into the input of one ℓ-ary operation. And a morphism $\eta\colon k \to \mathcal{P}(1)$ called the unit.

This composition is required to be associative – given three levels of operations, composing levels 1 and 2, then composing with level 3 is the same as first composing levels 2 and 3, then composing the result with level 1. To write this formally means requiring the following diagram to commute:

$$\begin{array}{ccc}
\mathcal{P}(n) \otimes (\bigotimes_{t=1}^{n} \mathcal{P}(i_t)) \otimes (\bigotimes_{r=1}^{i} \mathcal{P}(j_r)) & \xrightarrow{\gamma \otimes \mathrm{id}} & \mathcal{P}(i) \otimes (\bigotimes_{r=1}^{i} \mathcal{P}(j_r)) \\
 & & \downarrow \gamma \\
\cong \downarrow \text{shuffle} & & \mathcal{P}(j) \\
 & & \uparrow \gamma \\
\mathcal{P}(n) \otimes (\bigotimes_{t=1}^{n} (\mathcal{P}(i_t) \otimes (\bigotimes_{r=i_{t-1}+1}^{i_t} \mathcal{P}(j_r)))) & \xrightarrow{\mathrm{id} \otimes (\bigotimes_t \gamma)} & \mathcal{P}(n) \otimes (\bigotimes_{t=1}^{n} \mathcal{P}(h_t))
\end{array}$$

The unit axioms say the following two diagrams commute:

$$\begin{array}{ccc} \mathcal{P}(n)\otimes k^{\otimes n} & \xrightarrow{\cong} & \mathcal{P}(n) \\ {\scriptstyle \mathrm{id}\otimes\eta^{\otimes n}}\downarrow & \nearrow_{\gamma} & \\ \mathcal{P}(n)\otimes\mathcal{P}(1)^{\otimes n} & & \end{array} \quad \text{and} \quad \begin{array}{ccc} k\otimes\mathcal{P}(i) & \xrightarrow{\cong} & \mathcal{P}(i) \\ {\scriptstyle \eta\otimes\mathrm{id}}\downarrow & \nearrow_{\gamma} & \\ \mathcal{P}(1)\otimes\mathcal{P}(i) & & \end{array}$$

Notice η picks out a distinguished element $\eta(1_k) \in \mathcal{P}(1)$. The unit axioms amount to saying $\eta(1)$ is the identity operation. Composing with $\eta(1)$ returns the starting operation.

There is also the notion of a symmetric operad. This means that $\mathcal{P}(n)$ not just a vector space, but is in fact a representation of S_n; we can act on vectors in $\mathcal{P}(n)$ by permutations. This action of S_n should be though of as permuting the inputs to an n-ary function. It allows us to capture the symmetries of the n-ary operations.

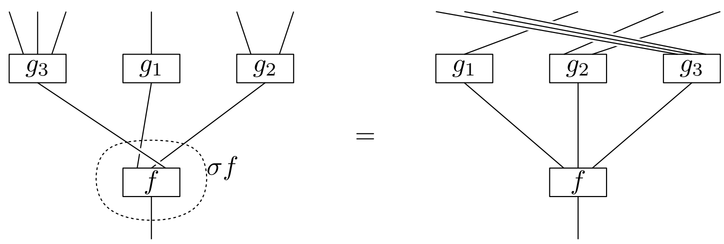
For example, suppose that in some symmetric operad $\mathcal{P}$, $\mu \in \mathcal{P}(2)$ is some sort of binary (multiplication) operation. Since $\mathcal{P}(2)$ is a representation of S_2, we have the notion of $(12)\cdot\mu$. We have that $((12)\cdot\mu)(a,b) = \mu(b,a)$, acting by (12) permutes the inputs according to the permutation. To say that μ is commutative means that $\mu(b,a) = \mu(a,b)$, so on the level of abstract operations $(12)\cdot\mu = \mu$. This can be restated as $\mathrm{span}\{\mu\}$ is the trivial representation of S_2.

On upgrading to symmetric operads, we have to add another axiom which says that the composition is equivariant with respect to the action of S_n. Formally it means that the following diagrams commute:

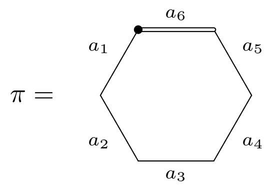
$$\begin{array}{ccc} \mathcal{P}(n)\otimes(\mathcal{P}(i_1)\otimes\cdots\otimes\mathcal{P}(i_n)) & \xrightarrow{\sigma\otimes\sigma^{-1}} & \mathcal{P}(n)\otimes(\mathcal{P}(i_{\sigma(1)})\otimes\cdots\otimes\mathcal{P}(i_{\sigma(n)})) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \mathcal{P}(i) & \xrightarrow{\sigma(i_{\sigma(1),\ldots,\sigma(n)})} & \mathcal{P}(i) \end{array}$$

$$\begin{array}{ccc} \mathcal{P}(n)\otimes(\mathcal{P}(i_1)\otimes\cdots\otimes\mathcal{P}(i_n)) & \xrightarrow{\mathrm{id}\otimes(\tau_1\otimes\cdots\otimes\tau_n)} & \mathcal{P}(n)\otimes(\mathcal{P}(i_{\sigma(1)})\otimes\cdots\otimes\mathcal{P}(i_{\sigma(n)})) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \mathcal{P}(i) & \xrightarrow{\tau_1\oplus\cdots\oplus\tau_n} & \mathcal{P}(i) \end{array}$$

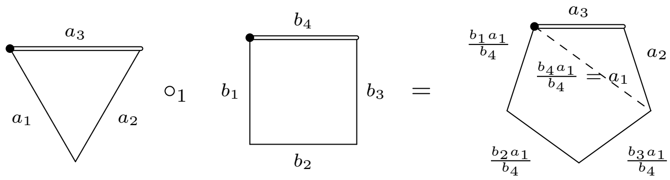
Of course it is not especially easy to interpret what these diagrams say. Tracing through the maps from the upper left, one sees that the idea of the composition being equivariant can be summed up nicely with the following schematic picture:



Now we can see how the gluing of polygons can give a operad structure, at least on the vector space of F^*-deco polygons. The root side of the polygon:



naturally distinguishes it from the other sides, and through operadic glasses one could view it as some sort of output. The remaining sides could be seen as inputs to π, and a composition could be obtained by gluing other polygons to them. The labels on the glued edges won't necessarily match; to remedy this, scale the gluing polygon so they do. For example gluing into the first slot, with the *partial composition* $\circ_1$:

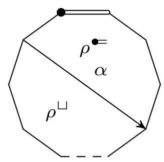


At least superficially, this idea of composition is very similar to the composition of polygons in the mosaic operad [14]. That operad consists of polygons, with marked diagonals. They are glued together along their sides, but the gluing is remembered as a diagonal in the new polygon. Since there is no distinguished root side, this operad is in fact a *cyclic operad* as defined by [21], in the sense that S_{n+1} can act on $\mathcal{P}(n)$ in such a way that inputs and output exchanged.

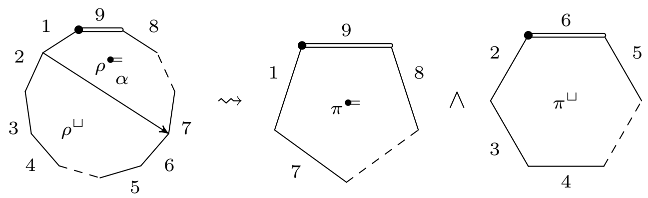
2.7.3 A VV-Differential on Polygons

I can also define another differential on the polygon algebra, distinctly different from the differentials used in either of [1, 18]. This differential is defined using VV-arrows going between two vertices of an R-deco polygon.

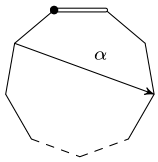
Given a vertex-vertex arrow on an R-deco polygon, the root region $\rho^{\bullet=}$ and cutoff region $\rho^{\sqcup}$ are defined as follows:

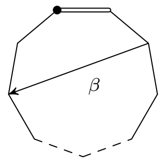


The induced polygons come from collapsing the arrow α, and choosing the first vertex of the cut-off polygon to be the vertex coming from the collapsed arrow. The root side of the cut-off polygon is chosen to be the edge pointed to by α in the cut-off region.



An arrow α is called backwards if it ends before it begins, using the ordering of vertices with the first vertex being $\bullet$, and the last vertex being the other vertex of the root side $=$.





On the left, α is forwards, and on the right β is backwards.

For an R-deco polygon π, define:

$$\partial_{\mathrm{VV}}\pi = \sum_{\substack{\alpha \text{ a VV-arrow} \\ \text{first vertex } \bullet \notin \alpha}} \mathrm{sgn}(\alpha)\pi^{\bullet\!=} \wedge \pi^{\sqcup}$$

where

$$\mathrm{sgn}(\alpha) = \begin{cases} 1 & \alpha \text{ forwards} \\ (-1)^{\#\text{ sides } \pi^{\sqcup}} & \alpha \text{ backwards} \end{cases}$$

and we do not bother with trivial arrows, those arrows going between adjacent vertices.

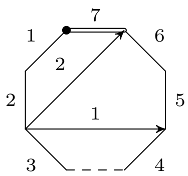
The differential ∂_{VV} is extended to the rest of the polygon algebra as expected using the Leibniz rule $\partial(a \wedge b) = \partial a \wedge b + (-1)^{\deg a} a \wedge \partial b$.

Proposition 2.5. *The construction ∂_{VV} is a differential: $\partial_{VV}^2 = 0$ on R-deco polygons.*

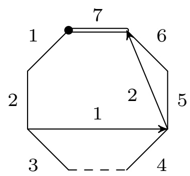
The proof of this will proceed by a series of lemma which show how different combinations of arrows can be made to cancel in the second derivative.

Firstly observe that terms in ∂^2 arise from 2-VV-dissections of the polygon π, where the arrows are given an ordering for which is cut off first. We then get the term in ∂^2 by cutting off each arrow in turn.

Notice that some of these 2-VV-dissections give exactly the same term in ∂^2 (the same in the sense that they make the same sequence of cuts, rather than just coincidentally give the same term). For example:



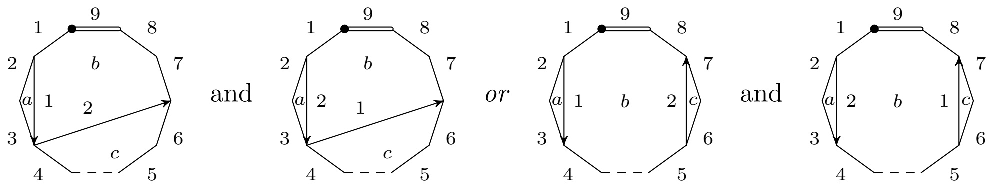
is the same sequence of cuts as



since the head and tail of arrow 1 are identified after cutting. This is the only way in which 'different' dissections can be equivalent, since the first arrow is obviously determined by the term it give in $\partial\pi$, and the second arrow has at most two pre-images in the original polygon. There is only one identified vertex in the quotient polygons, and so only one end of the second arrow is ambiguous. In this situation, we choose the configuration of arrows where the meeting ends are different – the right hand picture above.

Lemma 2.6. *If the dual tree of the dissection is non-linear $a - (b) - c$, then changing the order of the cut-off gives ters: $(b \wedge c) \wedge a$ and $(b \wedge a) \wedge c$, with the same sign.*

Proof. Pictorially it is clear the terms are as indicated.

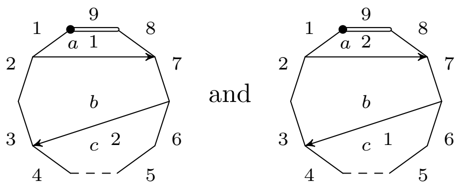


The second arrow always lies in the root region after cutting the first arrow. So the there is no complication with introducing a new root edge for the second arrow to interact with. The root edges and first vertices of the two cut-off regions can be read off directly from the diagram without needing to worry about the ordering of arrows. So we get the terms $(a1b2c) \rightsquigarrow (b|c) \wedge a \rightsquigarrow (b \wedge c) \wedge a$, and $(a2b1c) \rightsquigarrow (a|b) \wedge c \rightsquigarrow (b \wedge a) \wedge c$. Each time, the second cut happens in the first term, so by Leibniz both gain $+1$.

Since the second arrow lies in the root region after cutting the cut, it also means the direction of the second arrow is not changed, the vertices of the root region retain the same ordering. So the sign of the arrow is the same whether cut first or second. So we are just multiplying both possibilities by the same sign: 1 or $(-1)^{\#\text{ sides } a}$, then 1 or $(-1)^{\#\text{ sides } c}$. ☐

Lemma 2.7. *If the dual tree is linear $(a) - b - c$, but the arrows to not intersect anywhere (not even at the end points), then changing the cut-of order gives the same term with opposite signs.*

Proof. Pictorially it is also clear that the same terms appear. The arrows are far enough apart that the introduction of another root edge after the first cut does not cause a problem



We get $(a1b2) \rightsquigarrow a \wedge (b|c) \rightsquigarrow -a \wedge (b \wedge c)$, and $(a2b1c) \rightsquigarrow (a|b) \wedge c \rightsquigarrow (a \wedge b) \wedge c$.

When cutting the upper edge first, we pick up an extra minus sign from the Leibniz rule, since the second cut is pushed into the second factor of the wedge product.

However, we need to check that the remaining signs are correct. Notice that if the upper arrow is backwards, then when cutting it first the lower arrow reverses direction in the induced polygon. Check the possible combinations (in the table we include the sign coming from Leibniz):

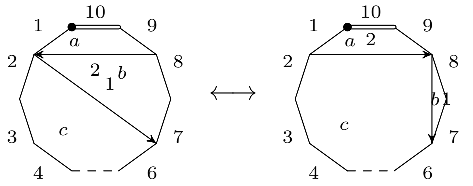
up	down	down after cut	parity from up then down	parity from down then up
fwd	fwd	fwd	$0+0+1$	$0+0$
fwd	bwd	bwd	$0+c+1$	$c+0$
bwd	fwd	bwd	$(b+c)+c+1 = b+1$	$0+b$
bwd	bwd	fwd	$(b+c)+1$	$c+b$

And indeed the signs are always opposite. ☐

Lemma 2.8. *If the dual tree of the dissection is linear, and the arrows intersect, then the second arrow will be nearest the root. By reversing it, and repositioning the first to keep the canonical form, we get terms $(a \wedge b) \wedge c$, and $(a \wedge c) \wedge b$, with the same sign.*

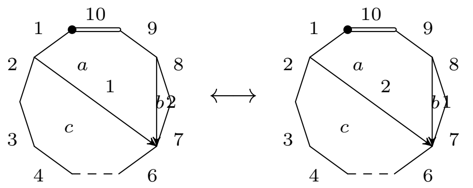
Proof. If the dual tree is linear and the arrows intersect, then we must cut the lower arrow first, otherwise the lower arrow will start at the root of the cut-off polygon. Hence the upper arrow is 2, and the lower arrow is 1.

Now reversing the upper arrow and repositioning gives us the following:



Both of these have the same type of dual tree, and arrow behaviour: they are linear and arrows intersect.

We can see that both of these give appropriate terms with the same signs by going to the non-cannonical choice of dissection. Swap which end of the first arrow the second arrow intersects. We get the same terms but with the following dissections:



So we are in fact just in the case where the dual tree is non-linear, and we have swapped the order of cut off. We know this gives the same sign and the terms as described above by the first lemma. ☐

Proof of Proposition. We have now established a pairing of all terms in ∂^2_{VV}. In each case we see the terms are equal but with opposite sign. Hence the terms cancel pairwise, and $\partial^2_{\mathrm{VV}}\pi = 0$ as claimed. ☐

This is probably not right way to go about introducing an extra grading by diagonals on the polygon algebra. However it's similarity to other constructions, like the Hopf algebra coproduct on Dupont's dissection polylogarithms [16], may shed some light on how to generalise the polygon algebra in this direction. Dupont defines a dissection diagram as a rooted polygon with diagonal arrows, and uses this to encode his dissection polylogarithm's. Terms in the coproduct for these dissection diagrams come from collapsing subsets of diagonal arrows in the diagram.

It has been suggested to me that polygons with diagonals seem related to blow-ups of curves and the boundary components of the moduli spaces $\mathfrak{M}_{0,n}$, defined in [10], this is also captured in the superficially similar mosaic operad. And that the dual tree of a fully triangulated polygon has some correspondence with maximally degenerate curves, defined in [34]. These glimpses of similar and related structures provide various new avenues to explore.

3 Multiple Zeta Values

3.1 Definitions

The multiple zeta values (henceforth abbreviated MZVs) are an intriguing class of numbers first studied by Euler in the special case of double zeta values (DZVs). The general case is introduced by Hoffman [29]. The multiple zeta function is a generalisation of the Riemann zeta function to a k-tuple of arguments, defined by:

$$\zeta(s_1, s_2, \ldots, s_k) := \sum_{0<n_1<n_2<\cdots<n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

Taking $s_1, s_2, \ldots, s_k$ to be integers in $\mathbb{Z}_{>0}$, we get the multiple zeta value $\zeta(s_1, s_2, \ldots, s_k)$. For this to be convergent we require $s_k > 1$.

Remark. There are two competing conventions about the index of summation. Some take $n_1 > n_2 > \cdots > n_k > 0$, rather than the index $0 < n_1 < n_2 < \ldots < n_k$ used above. This essentially has the effect of reversing the arguments to the multiple zeta function, and means convergence requires $s_1 > 1$. One must be aware of which convention is in use.

Notice that the multiple zeta value $\zeta(s_1, s_2, \ldots, s_k)$ is nothing more than a special value of the multiple polylogarithm $\mathrm{Li}_{s_1,s_2,\ldots,s_k}$:

$$\zeta(s_1, s_2, \ldots, s_k) = \mathrm{Li}_{s_1,s_2,\ldots,s_k}(1, 1, \ldots, 1)$$

Give a multiple zeta value $\zeta(s_1, s_2, \ldots, s_k)$, we call:

- The sum of its arguments $s_1 + s_2 + \cdots + s_k$ the weight, and
- The number of its arguments k the depth.

3.2 Relations and Transcendence

Much of the interest and work in MZVs concerns finding an proving identities between them, to understand all the relations betweem MZVs. Of particular interest are identities which give a MZV as a polynomial in single (Riemann) zeta values.

As an example of the sort of identities which arise when studying MZVs, we have Euler's famous identity:

$$\zeta(1, 2) = \zeta(3)$$

which is but the first instance of a result called *duality* of MZVs.

But we have plenty of other identities, like the Zagier-Broadhurst evaluation [Example 2.2 and Section 11 in 6]:

$$\zeta(\{1, 3\}^n) = \frac{1}{2n+1} \frac{\pi^{4n}}{(4n+1)!} = \frac{1}{2n+1} \zeta(2^{2n})$$

where here $\{1, 3\}^n$ is short-hand for the string $\{1, 3, 1, 3, \ldots, 1, 3\}$, with n copies of $1, 3$. Similarly 2^{2n}, which should be more properly written $\{2\}^{2n}$, is short-hand for $\{2, 2, \ldots, 2\}$, with n copies of 2.

And the Gangl-Kaneko-Zagier family of identities between DZV, which arise form a connection to *modular forms*, [19]. The first identity in this family is:

$$28\zeta(3, 9) + 150\zeta(5, 7) + 168\zeta(7, 5) = \frac{5197}{691} \zeta(12)$$

which occurs at $k = 12$, when there is a non-trivial *cusp form* of weight k on Γ_1.

As may be noticeable in the above examples, all known relations between MZVs break up into homogeneous pieces: the relations are between MZVs of the same weight. Conjecturally, all relations between MZVs are homogeneous, and so the vector space of MZVs is in fact weight graded. The Direct Sum Conjecture in [17] is essentially states:

Conjecture 3.1 (Direct Sum Conjecture)**.** *When regarded as a $\mathbb{Q}$-vector space, the space of MZVs is the direct sum of the subspaces $\mathcal{Z}_k$ of MZVs of weight k, so that all relations are homogeneous with respect to weight.*

The irrationality, transcendence and linear independence properties of these numbers are still very mysterious. Thanks to Euler we know that $\zeta(2k) \in \pi^{2k}\mathbb{Q}$, so that all even zetas are irrational and algebraically dependent. Moreover, since π is transcendental, they are linearly independent, but that's about all we know.

The only other explicit result on irrationality of MZVs is due to Apéry, as recently as 1978, when he proved that $\zeta(3)$ is irrational [2]. No one can even prove that $\zeta(5)$ is irrational, and aside from some curious non-explicit results like one of $\zeta(5), \zeta(7), \zeta(9)$ and $\zeta(11)$ is irrational, as are infinitely many $\zeta(\text{odd})$ [37], little more is known. The question of proving $\zeta(5)$ and $\zeta(3)$ are even linearly independent, i.e. $\zeta(5)/\zeta(3)$ is irrational, seems hopelessly out of reach.

Conjecture 3.2 (Algebraic Independence [Conjecture 1 in 46])**.** *The numbers $\pi, \zeta(3), \zeta(5), \zeta(7), \zeta(9), \ldots$, are algebraically independent over $\mathbb{Q}$.*

Following extensive numerical computations, searching for linear relations between MZVs, Zagier found numerically that the dimension of the space $\mathcal{Z}_k$ of MZVs of weight k is given by:

k	2	3	4	5	6	7	8	9	10	11	12
$\dim_{\mathbb{Q}} \mathcal{Z}_k$	1	1	1	2	2	3	4	5	7	9	12

This leads to the general conjecture [Section 9 in 42] that $\dim_{\mathbb{Q}} \mathcal{Z}_k$ is given by the coefficient of x^k in the expansion of $\frac{1}{1-x^2-x^3}$, or equivalently by d_k, where d_k is defined by the recurrence relation:

$$\begin{cases} d_k = d_{k-2} + d_{k-3} \text{ with} \\ d_2 = d_3 = d_4 = 1 \end{cases}$$

Conjecture 3.3 (Dimension Conjecture)**.** *The dimension of the space $\mathcal{Z}_k$ of MZVs of weight k is given by d_k, satisfying the recurrence $d_k = d_{k-2} + d_{k-3}$ with initial conditions $d_2 = d_3 = d_4 = 1$.*

This recurrence relating weight k MZVs to weight $k-2$ and weight $k-3$ in turn lead Hoffman to propose a candidate basis for the space $\mathcal{Z}_k$ might be given by $\zeta(w)$, where the word w is of weight k and satisfies $w \in \{2,3\}^\times$. That is a basis might consist of zetas where the arguments are 2's and 3's only, [Conjecture C in 30].

Conjecture 3.4 (Basis Conjecture)**.** *A basis for the space $\mathcal{Z}_k$ is given by the Hoffman elements $\zeta(n_1, \ldots, n_r)$, where $n_1, \ldots, n_r \in \{2,3\}$, with weight k.*

Of course these *numerical* computations in no way establish any bound on the dimension. Any numerical relations found agree only to the precision each side is calculated, they may differ when computed to higher precision. Lack of a numerical relation just means there any relation which holds has rational factors whose denominators are larger than the working precision, and so not identifiable.

It has since been prove, by various authors such as Goncharov [23], Terasoma [38] and Brown [8], that the upper bound $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$ indeed holds. It is also known from Brown's work that the Hoffman elements, ζ(words in 2's and 3's of weight k), do *span* the space of MZVs. I can sketch some ideas from one proof of this which uses Brown's motivic MZVs, to be introduced later.

The reverse inequality is *much* harder to tackle. We don't even have a single proven instance where $\dim_{\mathbb{Q}} \mathcal{Z}_k > 1$. Nobody seriously entertains the notation that $\dim_{\mathbb{Q}} \mathcal{Z}_k = 1$, but for all we know the MZVs of weight k are all rational multiples of $\zeta(k)$, with immensely complicated rational factors we haven't identified yet.

3.3 Algebraic Structure of MZVs and the Standard Relations

Using the point of view that MZVs are special values of the multiple polylogarithm, and using the integral representation of the multiple polylogarithm, we obtain the Kontsevich representation of an MZV:

$$\zeta(n_1, n_2, \ldots, n_k) = (-1)^k I(0; 10^{n_1-1}\, 10^{n_2-1} \,\cdots\, 10^{n_k-1}; 1)$$

This motivates encoding an MZV $\zeta(n_1, n_2, \ldots, n_k)$ as a word $yx^{n_1-1}yx^{n_2-1}\cdots yx^{n_k-1}$ in the non-commutative polynomial ring $\mathbb{Q}\langle x, y\rangle$, as in [28] from which the following comes. Such a word corresponds to a convergent MZV if and only if it begins with y and ends in x, these are the <u>admissible</u> words. Denote by $\mathfrak{H}^0$ the vector space of admissible words, we then view ζ as a $\mathbb{Q}$-linear map $\zeta\colon \mathfrak{H}^0 \to \mathbb{R}$. This encoding puts MZVs on much more algebraic footing, and allows very elegant statements of relations.

With this we can formulate the first family of relations.

Duality: Define the antiautomorphism $\tau\colon \mathbb{Q}\langle x, y\rangle \to \mathbb{Q}\langle x, y\rangle$ by $\tau(x) = y$, and $\tau(y) = x$. This then simultaneously reverses a word and interchanges $x \leftrightarrow y$. Notice that this preserves the admissibility of words. The duality theorem then states:

$$\zeta(w) = \zeta(\tau(w))$$

for any admissible word $w \in \mathfrak{H}^0$.

For $w = yx^{n_1-1}yx^{n_2-1}\cdots yx^{n_k-1}$, this is essentially proven by considering the integral representation:

$$\zeta(w) = (-1)^k I(0; 10^{n_1-1}\, 10^{n_2-1} \cdots 10^{n_k-1}; 1)$$

Then apply the change of variables $t_i' = 1 - t_i$ to the iterated integral, to arrive at:

$$\begin{aligned} &= (-1)^{k'} I(0; 1^{n_k-1}0\, 1^{n_{k-1}-1}0 \cdots 1^{n_1-1}0; 1) \\ &= \zeta(\tau(w)) \end{aligned}$$

since the word has been reversed, and had $0 \leftrightarrow 1$ interchanged.

From this one gets a proof of Euler's identity. Consider $\zeta(3)$. We have 3 is encoded by $w = yx^2$. But $\tau(w) = y^2x$, which encodes $\{1, 2\}$. So $\zeta(3) = \zeta(1, 2)$.

Without this algebraic encoding of MZVs, formulating the statement of duality is a much more awkward prospect. The integral representation of MZVs makes the proof almost trivial.

Shuffle product: We may find the product of two MZVs by multiplying their integral representations. We know that iterated integrals multiply by the shuffle product, so we can define a shuffle product on $\mathbb{Q}\langle x, y\rangle$, which reflects this. The shuffle product $\sqcup\!\sqcup$ is computed recursively by:

- For any word w, $1 \sqcup\!\sqcup w = w \sqcup\!\sqcup 1 = w$, where 1 is the empty word.
- For any words w_1, w_2, and symbols $a, b \in \{x, y\}$:

$$aw_1 \sqcup\!\sqcup bw_2 = a(w_1 \sqcup\!\sqcup bw_2) + b(aw_1 \sqcup\!\sqcup w_2)$$

This endows $(\mathbb{Q}\langle x, y\rangle, \sqcup\!\sqcup)$ with the structure of a commutative algebra. The fact that iterated integrals multiply with the $\sqcup\!\sqcup$-product says that $\zeta\colon (\mathfrak{H}^0, \sqcup\!\sqcup) \to (\mathbb{R}, \cdot)$ is a homomorphism, meaning:

$$\zeta(w_1)\zeta(w_2) = \zeta(w_1 \sqcup\!\sqcup w_2)$$

This gives us another family of relations between MZVs.

For example, $\zeta(2) = \zeta(yx)$. We have:

$$\zeta(2)\zeta(2) = \zeta(yx)\zeta(yx) = \zeta(yx \sqcup\!\sqcup\, yx)$$

We readily compute the shuffle product to be $2 \cdot yxyx + 4 \cdot yyxx$, so that:

$$\zeta(yx \sqcup\!\sqcup\, yx) = \zeta(2 \cdot yxyx + 4 \cdot yyxx) = 2\zeta(yxyx) + 4\zeta(yyxx) = 2\zeta(2,2) + 4\zeta(1,3)$$

giving:

$$\zeta(2)\zeta(2) = 2\zeta(2,2) + 4\zeta(1,3)$$

Stuffle product: We can also multiply MZVs by multiplying their series representations. The product of two such series is a sum of series where the indices of summation are taken in all possible ways compatible with the original indices.

For example:

$$\begin{aligned}\zeta(2)\zeta(2) &= \sum_{n>0} \frac{1}{n^2} \sum_{m>0} \frac{1}{m^2} \\ &= \sum_{n>m>0} \frac{1}{m^2 n^2} + \sum_{m>n>0} \frac{1}{n^2 m^2} + \sum_{n=m>0} \frac{1}{n^2 m^2} \\ &= \zeta(2,2) + \zeta(2,2) + \zeta(4) \\ &= 2\zeta(2,2) + \zeta(4)\end{aligned}$$

This is reflected in the stuffle product on $\mathbb{Q}\langle x, y\rangle$. The stuffle product $*$ is computed recursively by:

- For any word w, $1 * w = w * 1 = w$,
- For any word w, and any integer $n \geq 1$:

$$x^n * w = w * x^n = wx^n$$

- For any words w_1, w_2, and integers $p, q \geq 0$:

$$yx^p w_1 * yx^q w_2 = yx^p(w_1 * yx^q w_2) + yx^q(yx^p w_1 * w_2) + yx^{p+q+1}(w_1 * w_2)$$

This last requirement reflects the fact that when multiplying series, the argument strings of the MZVs are shuffled in all possible ways (the first two terms), and that two arguments can be *stuffed* into the same slot (the third term).

This endows $(\mathbb{Q}\langle x, y\rangle, *)$ with a different commutative algebra structure. The fact that these series multiply with $*$-product says that $\zeta\colon (\mathfrak{H}^0, *) \to (\mathbb{R}, \cdot)$ is a homomorphism, meaning:

$$\zeta(w_1)\zeta(w_2) = \zeta(w_1 * w_2)$$

This is a third family of relations between MZVs.

(Regularised) Double Shuffle: We have two distinct ways of multiplying MZVs: they cry out to be compared. The two multiplications above give us quadratic relations between MZVs, but when comparing the results, we now get *linear* relations between MZVs.

For example, with the two different expressions for $\zeta(2)\zeta(2)$ above:

$$\begin{aligned}\zeta(2)\zeta(2) &= 2\zeta(2,2) + 4\zeta(1,3) \\ &= 2\zeta(2,2) + \zeta(4)\end{aligned}$$

we deduce:

$$2\zeta(2,2) + 4\zeta(1,3) = 2\zeta(2,2) + \zeta(4) \implies 4\zeta(1,3) = \zeta(4)$$

More generally we have the standard family of linear relations:

$$\zeta(w_1 * w_2 - w_1 \sqcup\!\sqcup\, w_2) = 0$$

for any $w_1, w_2 \in \mathfrak{H}^0$.

Unfortunately this family of relations is known to be insufficient for generating *all* linear relations between MZVs. For this reason we introduce a formal symbol $\zeta(1)$ for the divergent MZV, and extend the map ζ to certain non-admissible words. Comparing shuffle and stuffle here leads to all divergent terms (formally) cancelling, and new linear relations appearing. The cancellation happens in such a way as to give valid results.

For example:

$$\zeta(2)\zeta(1) = \zeta(2 * 1) = \zeta(2,1) + \zeta(1,2) + \zeta(3) \text{ and}$$
$$\zeta(2)\zeta(1) = \zeta(yx \sqcup\!\sqcup\, y) = \zeta(yxy + 2 \cdot yyx) = \zeta(2,1) + 2\zeta(1,2)$$

thus:

$$\zeta(2,1) + \zeta(1,2) + \zeta(3) = \zeta(2,1) + 2\zeta(1,2) \implies \zeta(3) = \zeta(1,2)$$

giving another proof of Euler's identity.

Conjecturally, all relations between MZVs come from this regularised comparison of shuffle and stuffle. Notice that all relations arising in this way are necessarily homogeneous. Zudilin phrases this as:

Conjecture 3.5 (MZV Relations [Conjecture 2 in 46])**.**

$$\ker \zeta = \{u \sqcup\!\sqcup\, v - u * v \mid u \in \mathfrak{H}^1, v \in \mathfrak{H}^0\}$$

where $\mathfrak{H}^1 \coloneqq \mathbb{Q}1 + y\mathbb{Q}\langle x, y\rangle$*, corresponding to the inclusion of words not ending in* x*, which equate to divergent MZVs* $\zeta(n_1, n_2, \ldots, n_k)$ *with* $n_k = 1$*.*

3.4 Motivic Multiple Zeta Values

We have already encountered Goncharovs motivic iterated integrals. If we have known about MZVs at this point, we could have defined his version of motivic MZVs. We would take:

$$\zeta^{\mathcal{M}}(n_1, n_2, \ldots, n_k) \coloneqq (-1)^k I^{\mathcal{M}}(0; 10^{n_1-1}\, 10^{n_2-1} \cdots 10^{n_k-1}; 1)$$

in analogy with the Kontsevich integral representation.

On the motivic level questions about the linear independence of the motivic elements $\zeta^{\mathcal{M}}(2k+1)$ become trivial. Indeed, since $A_\bullet(\mathbb{Q})$ is weight graded, and the elements $\zeta^{\mathcal{M}}(2k+1)$ lie in components of different degree, they are linearly independent! Similarly for the question of whether all relations are homogeneous. This is what I intimated earlier when saying the motivic framework eliminates transcendence problems of classical iterated integrals.

One unsatisfactory aspect of Goncharov's motivic MZVs comes from the value of $\zeta^{\mathcal{M}}(2k)$. Since $(2\pi \mathrm{i})^{-2k}\zeta(2k) \in \mathbb{Q}$ by Euler, Goncharov says that we have $\zeta^{\mathcal{M}}(2k) = 0$. Brown shows how these can be further lifted in such a way that $\zeta^{\mathfrak{m}}(2)$ is non-zero. Details for this section are found in [8]

For parameters $a_i \in \{0,1\}$, the motives corresponding to Goncharov's motivic iterated integrals $I^{\mathcal{M}}(a_0; a_1, \dots, a_n; a_{n+1})$ are unramified over $\mathbb{Z}$, so they lie in $\mathcal{A}^{\mathcal{MT}} := \mathcal{A}_{\bullet}(\mathbb{Q})$. Introduce a trivial comodule over $\mathcal{A}^{\mathcal{MT}}$ defined by:

$$\mathcal{H}^{\mathcal{MT}_+} := \mathcal{A}^{\mathcal{MT}} \otimes_{\mathbb{Q}} \mathbb{Q}[f_2]$$

where f_2 is taken to be of degree 2. This will correspond to the non-zero lifting of $\zeta^{\mathcal{M}}(2)$.

Brown proves that there is a sub-Hopf algebra $\mathcal{A}$ of $\mathcal{A}^{\mathcal{MT}}$, and a graded comodule $\mathcal{H}$ over $\mathcal{A}$ satisfying the following properties:

It is spanned by the motivic iterated integrals:

$$I^{\mathfrak{m}}(a_0; a_1, \dots, a_n, a_{n+1}) \in \mathcal{H}_n$$

with $a_i \in \{0,1\}$, and satisfying the standard properties of iterated integrals given in subsection 2.4.

There is a period map:

$$\begin{gathered}\mathrm{per}\colon \mathcal{H} \to \mathbb{R} \\ I^{\mathfrak{m}}(a_0; a_1, \dots, a_n; a_{n+1}) \mapsto I(a_0; a_1, \dots, a_n; a_{n+1})\end{gathered}$$

which is a ring homomorphism, so motivic relations descend to classical iterated integrals.

There is a non-canonical isomorphism:

$$\mathcal{H} \cong \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta^{\mathfrak{m}}(2)]$$

and an embedding of algebra-comodules $\mathcal{H} \hookrightarrow \mathcal{H}^{\mathcal{MT}_+}$ which sends $\zeta^{\mathfrak{m}}(2)$ to f_2. Here $\zeta^{\mathfrak{m}}(2) := -I^{\mathfrak{m}}(0; 10; 1)$, and is non-zero in this incarnation. Goncharov's motivic MZVs are recovered by the quotient map $\mathcal{H} \to \mathcal{A}$ killing $\zeta^{\mathfrak{m}}(2)$. Brown denotes the image of $I^{\mathfrak{m}}$ under this quotient by $I^{\mathfrak{a}}$, and similarly for motivic MZVs.

On $\mathcal{H}$ we get a coaction $\Delta\colon \mathcal{H} \to \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$, which is computed by the same formula as Goncharov's coproduct, with the factors swapped:

$$\Delta I^{\mathfrak{m}}(a_0; a_1, \dots, a_n; a_{n+1}) = \sum_{0 < i_0 < i_1 < \dots < i_k < i_{k+1} = n+1} \left(\prod_{p=0}^{k} I^{\mathfrak{a}}(a_{i_p}; a_{i_p+1}, \dots, a_{i_{p+1}-1}; a_{i_{p+1}}) \right) \otimes I^{\mathfrak{m}}(a_0; a_{i_1}, \dots, a_{i_k}; a_{n+1})$$

To simplify the formula for the coaction, to make calculations more tractable, Brown wants to consider an infinitesimal version of it. Consider projection of $\mathcal{A}_{>0}$ to the Lie algebra of indecomposables:

$$\pi\colon \mathcal{A}_{>0} \to \mathcal{L} := \frac{\mathcal{A}_{>0}}{\mathcal{A}_{>0}\mathcal{A}_{>0}}$$

The image of an element $I^{\mathfrak{m}}$ under the projection to $\mathcal{L}$ is denoted by Brown as $I^{\mathfrak{L}}$.

For each odd $r \geq 3$, he defines the operator:

$$D_r\colon \mathcal{H}_N \xrightarrow{\Delta_{r,N-r}} \mathcal{A}_r \otimes_{\mathbb{Q}} \mathcal{H}_{N-r} \xrightarrow{\pi \otimes \mathrm{id}} \mathcal{L}_r \otimes_{\mathbb{Q}} \mathcal{H}_{N-r}$$

Its action on the motivic iterated integral $I^{\mathfrak{m}}(a_0; a_1, \dots, a_n; a_{n+1})$ is given explicitly by:

$$D_r I^{\mathfrak{m}}(a_0; a_1, \dots, a_n; a_{n+1}) = \sum_{p=0}^{n-r} I^{\mathfrak{L}}(a_p; a_{p+1}, \dots, a_{p+r}; a_{p+r+1}) \otimes I^{\mathfrak{m}}(a_0; a_1, \dots, a_p, a_{p+1}, \dots, a_n; a_{n+1})$$

This can be interpreted as cutting off one segment of length r from a semicircular polygon, rather than all possible combinations of segments. The sequences $(a_p; a_{p+1}, \ldots, a_{p+r}; a_{p+r+1})$ are called subsequences and $(a_0; a_1, \ldots, a_p, a_{p+r}, \ldots, a_n; a_{n+1})$ are the quotient sequences.

The operators D_r, for r odd, form an integral part of an algorithm which can decompose a motivic MZV into a chosen basis, [9]. The operator D_r is used to extract the coefficient of $\zeta^{\mathfrak{m}}(r)$ as a polynomial in motivic MZVs. By decomposing all the sub and quotient sequences of lower weight in this basis, one can extract the coefficients of each basis element, and decompose the given motivic MZV.

The upshot of this framework is that if all the operators D_r, for odd $r < N$ vanish on a given combination of motivic MZVs of weight N, then this combination is a rational multiple of $\zeta^{\mathfrak{m}}(N)$, [Theorem 3.3 in 8]. These results descend to classical MZVs using the period map; this gives us some very powerful and simple combinatorial tools for establishing results about MZVs.

This can be seen something of a Galois theory for transcendental numbers – an elements behaviour under certain operators (0 under the D_{2k+1} verses invariance under field automorphisms) gives information about what form it has (rational multiple of $\zeta^{\mathfrak{m}}(N)$ verses lying in the base field). The operators D_r also exist for even r, but play no clear role in this framework. Might they somehow be used to extract the coefficient $\zeta^{\mathfrak{m}}(N)$, which otherwise remains elusive?

3.5 Dimension of MZVs and the Hoffman Elements

One of the first significant applications of Brown's motivic MZV framework is to give an alternative proof that the dimension of the space of MZV of weight k is bounded above by d_k, and that the Hoffman elements $\zeta(2\text{'s and }3\text{'s})$ span this space. Complete details are found in [8, 9].

The following is the combination of Lemma 3.3 and Remark 3.7 in [9]. By the period map, and by construction of $\mathcal{H} \hookrightarrow \mathcal{H}^{\mathcal{MT}_+}$, we have:

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \leq \dim_{\mathbb{Q}} \mathcal{H}_k \leq \dim_{\mathbb{Q}} \mathcal{H}_k^{\mathcal{MT}+}$$

By computing the Poincaré series we will determine $\dim_{\mathbb{Q}} \mathcal{H}_k^{\mathcal{MT}_+} = d_k$. Brown says that $\mathcal{A}^{\mathcal{MT}}$ is non-canonically isomorphic to the cofree Hopf algebra on cogenerators f_{2r+1} in degree $2r+1 \geq 3$, so that the comodule has the following structure:

$$\mathcal{H}^{\mathcal{MT}_+} \cong \mathbb{Q}\langle f_3, f_5, \ldots, f_{2r+1}, \ldots \rangle \otimes_{\mathbb{Q}} \mathbb{Q}[f_2]$$

The Poincaré series for $\mathbb{Q}\langle f_3, f_5, \ldots \rangle$ is given by:

$$\frac{1}{1 - t^3 - t^5 - \cdots - t^{2r+1} - \cdots} = \frac{1 - t^2}{1 - t^2 - t^3}$$

multiplying this by the Poincaré series for $\mathbb{Q}[f_2]$, which is $\frac{1}{1-t^2}$ gives the Poincaré series for $\mathcal{H}^{\mathcal{MT}_+}$ as:

$$\sum_{k \geq 1} \dim_{\mathbb{Q}} \left(\mathcal{H}_k^{\mathcal{MT}_+} \right) t^k = \frac{1}{1-t^2} \frac{1 - t^2}{1 - t^2 - t^3} = \frac{1}{1 - t^2 - t^3}$$

and $\dim_{\mathbb{Q}} \mathcal{H}_k^{\mathcal{MT}_+} = d_k$ as required

So the upper bound $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$ of Zagier's dimension conjecture holds.

In considering the elements $\zeta^{\mathfrak{m}}(2\text{'s and }3\text{'s})$, Brown is able to show they are linearly independent, [Theorem 7.4 in 8]. Their number in weight k is d_k, so gives the lower bound $\dim_{\mathbb{Q}} \mathcal{H}_k \geq d_k$ on the space of motivic iterated integrals of weight k. Overall this establishes an isomorphism $\mathcal{H} \cong \mathcal{H}^{\mathcal{MT}_+}$, not just an embedding.

With this he settles one conjecture about the structure of the *motivic Galois group* $\mathcal{G}_{\mathcal{MT}'}$ of $\mathcal{MT}'(\mathbb{Z})$. Here $\mathcal{MT}'(\mathbb{Z})$ is the full Tannakian subcategory of $\mathcal{MT}(\mathbb{Z})$ generated by the motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and $\mathcal{MT}(\mathbb{Z})$ is the category of mixed Tate motives unramified over $\mathbb{Z}$. The conjecture is that the map $\mathcal{G}_{\mathcal{MT}} \twoheadrightarrow \mathcal{G}_{\mathcal{MT}'}$ is an isomorphism, where $\mathcal{G}_{\mathcal{MT}}$ is the motivic Galois group of $\mathcal{MT}(\mathbb{Z})$. A further consequence of this is that the periods of $\mathcal{MT}(\mathbb{Z})$, of mixed Tate motives unramified over $\mathbb{Z}$, are $\mathbb{Q}[\frac{1}{2\pi \mathrm{i}}]$-linear combinations of MZVs.

The linear independence of $\zeta^{\mathfrak{m}}$(2's and 3's), and the number $d_k$of them in each weight k, means they form a basis for the space of motivic MZVs of weight k. So every motivic MZV can be written as a unique $\mathbb{Q}$-linear combination of these motivic Hoffman elements. Applying the period map shows that the elements Hoffman elements ζ(2's and 3's) *span* the space of classical MZVs, confirming one part of Hoffmans proposed basis conjecture.

Brown's proof that $\zeta^{\mathfrak{m}}$(2's and 3's) are linear independent works inductively on the level, defined to be the number of 3s in the word w of the argument of $\zeta^{\mathfrak{m}}$. The base case is provided by the fact that all Hoffman MZVs of level 0, i.e. the elements $\zeta^{\mathfrak{m}}(2^n)$, are linearly independent. The induction assumption is that all Hoffman MZVs of level $< \ell$ are linearly independent. Brown shows how a relation between Hoffman MZVs of level ℓ must imply a relation between Hoffman MZVs of strictly smaller level, which contradicts the induction assumption.

Establishing this relies heavily on an explicit computation of $\zeta(2, \ldots, 2, 3, 2, \ldots, 2)$ by Zagier [43], and the 2-adic properties of coefficients in this expansion.

3.6 Results using Motivic MZVs

In Equation 18 of [4], Borwein, Bradley, and Broadhurst conjecture that the following MZV is an (explicit) rational multiple of π^{wt}:

$$\zeta(\{2^m, 1, 2^m, 3\}^n, 2^m) = \frac{1}{2n+1} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

where I write wt to mean the weight of the MZV, here $(4 + 4m)n + 2m$.

Within Brown's motivic MZV framework it is surprisingly easy for me to establish the non-explicit version of this result: it is some rational multiple of π^{wt}. Although this comes with the caveat that the rational factor is not determined explicitly.

Proposition 3.6. *The MZV* $\zeta(\{2^m, 1, 2^m, 3\}^n, 2^m)$ *is a rational multiple of* π^{wt}.

Proof (Sketch)**.** First lift this to the motivic level, write out the binary string encoding this MZV, and notice how it can be written in a very symmetrical way as:

$$(01)^{m+1} \mid (10)^{m+1} \mid \cdots \mid (01)^{m+1}$$

where the bars | are purely to aid in seeing the pattern.

We want to show that D_{2k+1} vanishes completely on this iterated integral. The terms of D_{2k+1} correspond to ways of cutting out a subsequence of length $2k + 1$ from this string.

A non-trivial subsequence of odd length will lie in an even number of blocks, otherwise its start and end digit are equal, and it vanishes. By reflecting this sequence of blocks we define another subsequence of the string, which is the reverse of this subsequence. Reflecting is an involution on subsequences so we establish a pairing of subsequences.

The quotient sequences for these two subsequences agree, and by reversal of paths the iterated integrals defined by the subsequences are opposites. Hence these two terms cancel pairwise in D_{2k+1}. □

With this understood, it is really no more difficult to prove the generalisation that one may insert symmetrically all permutations of fixed blocks $2^{a_0}, 2^{a_1}, \ldots, 2^{a_{2n}}$ into the gaps of the string $\{1,3\}^n$, and obtain a rational multiple of π^{wt}.

Proposition 3.7 (Symmetric Insertion). *Fix $2n+1$ non-negative integers $a_0, a_1, \ldots, a_{2n} \geq 0$, and define:*

$$Z(a_0, a_1, \ldots, a_2 n) \coloneqq \zeta(2^{a_0}, 1, 2^{a_1}, 3, 2^{a_2}, \ldots, 1, 2^{a_{2n-1}}, 3, 2^{a_{2n}})$$

obtained by inserting the block 2^{a_i} into the i-th gap of the of the string $\{1,3\}^n$. Then:

$$\sum_{\sigma \in S_{2n+1}} Z(a_{\sigma(0)}, a_{\sigma(1)}, \ldots, a_{\sigma(2n)}) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

where S_{2n+1} is the symmetric group on $2n+1$ letters.

Proof. In this case we get strings like $(01)^{a_0+1} \mid (10)^{a_1+1} \mid \cdots \mid (01)^{a_{2n}+1}$, and all permutations. The reversing blocks procedure again sets up a pairwise cancellation between terms of D_{2k+1}, an the result follows. ☐

More generally, with the notation Z from the proposition above, the cyclic insertion conjecture formulated as Conjecture 1 in [5] by Borwein et al. proposes that:

$$\sum_{r \in C_{2n+1}} Z(a_{r(0)}, a_{r(1)}, \ldots, a_{r(2n+1)}) = \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

independently of the number of blocks or their individual sizes. Here C_{2n+1} is the cyclic group of order $2n+1$, and it acts naturally by cyclicly shifting the indices. So only inserting cyclic rotations of the blocks should be sufficient to get a rational multiple of π^{wt}.

The nearest result to this at the moment seems to be the Bowman-Bradley theorem which establishes an exotic shuffle relation between $\zeta(2^m)$ and $\zeta(\{1,3\}^n)$:

Theorem 3.8 (Bowman-Bradley [Theorem 5.1 in 7]).

$$\zeta(2^m \sqcup \{1,3\}^n) = \sum_{\substack{a_0+a_1+\cdots+a_{2n}=m \\ a_0,a_1,\ldots,a_{2n}\geq 0}} Z(a_0, a_1, \ldots, a_{2n}) = \binom{2n+m}{m} \frac{1}{2n+1} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

This says that one can insert all compositions (which may include 0 summands) of blocks with total size m to get a rational multiple of π^{wt}.

Notice that since any cyclic rotation of a composition $a_0 + a_1 + \cdots + a_{2n} = m$ still gives a composition of m, the cyclic insertion conjecture would imply this sum breaks up into π^{wt}-rational subsums over cyclic rotations of fixed compositions. The symmetric insertion result above tells us, at the first stage, this sum *does* indeed break up into π^{wt}-rational subsums over all *permutations* of fixed compositions.

A proof of a non-explicit cyclic insertion conjecture does not yet come easily with motivic MZVs – the binary encodings are not symmetrical enough to allow the sort of pairwise cancellation used significantly in the earlier proofs. Nevertheless we can try and see what happens.

Example 3.9. Looking at the example $\zeta(1,3,1,2,3,2)$ plus cyclic shifts. This gives the motivic iterated integrals

$$\begin{gathered} I^{\mathfrak{m}}(01 \mid 10 \mid 01 \mid 1010 \mid 0101) \\ I^{\mathfrak{m}}(01 \mid 10 \mid 0101 \mid 1010 \mid 01) \\ I^{\mathfrak{m}}(01 \mid 1010 \mid 0101 \mid 10 \mid 01) \\ I^{\mathfrak{m}}(0101 \mid 1010 \mid 01 \mid 10 \mid 01) \\ I^{\mathfrak{m}}(0101 \mid 10 \mid 01 \mid 10 \mid 0101) \end{gathered}$$

Computing D_3, and killing all the terms which obviously cancel, and making the appropriate simplifications, we are left with:

$$6\zeta^{\mathfrak{L}}(3) \otimes (\zeta^{\mathfrak{m}}(3,2,1,3) - \zeta^{\mathfrak{m}}(3,1,3,2) + \zeta^{\mathfrak{m}}(1,2,3,3) - \zeta^{\mathfrak{m}}(1,3,3,2))$$

which does not obviously cancel.

Applying the period map, and numerically evaluating the right hand side, we find:

$$\zeta(3,2,1,3) - \zeta(3,1,3,2) + \zeta(1,2,3,3) - \zeta(1,3,3,2) \approx 0$$

to several hundred decimal places. So on the motivic level this conceivably could be exactly 0, however I don't see this yet.

Similar problems happen for the other derivations D_r, and in other cases – the terms don't all cancel cleanly. But applying the period map seems always to lead to a result on classical MZVs which is 0, consistent with the operators D_r vanishing on the motivic MZVs. By keeping careful track of where the terms in these results come from, and how they seem to cancel, it appears a if the terms from a fixed position in a cyclicly shifted block sum to zero. This leads me to propose:

Conjecture 3.10 (Dihedral Insertion)**.** *Fix $2n$ non-negative integers $a_1, a_2, \ldots, a_{2n}$.*

Let $X_i(a_1, a_2, \ldots, a_{2n})$ be the MZV obtained by replacing the i-th 1 of the string $\{1,3\}^n$, with 2^{a_1}, and cyclicly inserting the other 2^{a_i} in the gaps to the right. Let $Y_i(a_1, a_2, \ldots, a_{2n})$ be the MZV obtained by replacing the i-th 1 with 2^{a_1}, and inserting the rest cyclicly in the gaps to the left.

Then:

$$\sum_{i=1}^{n} X_i(a_1, \ldots, a_{2n}) - Y_i(a_1, \ldots, a_{2n}) \stackrel{?}{=} 0$$

The name comes from this results similarity to cyclic insertion, but for the additional sign which arises when moving in reverse.

For example, with $n = 2$, and $a_1 = a$, $a_2 = b$, $a_3 = c$, $a_4 = d$, we could construct:

$$\begin{aligned} X_1(a,b,c,d) &= \zeta(\mathbf{2^{a}}, 3, 2^b, 1, 2^c, 3, 2^d) \\ Y_1(a,b,c,d) &= \zeta(\mathbf{2^{a}}, 3, 2^d, 1, 2^c, 3, 2^b) \\ X_2(a,b,c,d) &= \zeta(2^c, 1, 2^d, 3, \mathbf{2^{a}}, 3, 2^b) \\ Y_2(a,b,c,d) &= \zeta(2^c, 1, 2^b, 3, \mathbf{2^{a}}, 3, 2^d) \end{aligned}$$

and get:

$$\begin{aligned} &\zeta(2^a, 3, 2^b, 1, 2^c, 3, 2^d) - \zeta(2^a, 3, 2^d, 1, 2^c, 3, 2^b) \\ &\quad + \zeta(2^c, 1, 2^d, 3, 2^a, 3, 2^b) - \zeta(2^c, 1, 2^b, 3, 2^a, 3, 2^d) \stackrel{?}{=} 0 \end{aligned}$$

with the previous result being $a = b = c = 0, d = 1$.

3.7 Motivic DZVs and Eisenstein Series

As another example of how powerful the combinatorial tools of motivic MZVs are, I can derive a non-explicit version of the Gangl-Kaneko-Zagier identities between DZVs simply by considering the action of D_{2k+1} and using some linear algebra.

Consider the odd-odd motivic DZV:

$$\zeta^{\mathfrak{m}}(2a+1, 2b+1) \coloneqq I^{\mathfrak{m}}(0; 10^{2a}\, 10^{2b}; 1)$$

By looking at the result cutting out a subsequence of length $2k+1$ in the various cases $k < a$, $k = a$, and $k > a$, and similarly for b, we find that generally $D_{2k+1}\zeta^{\mathfrak{m}}(2a+1, 2b+1)$ is given by:

$$D_{2k+1}\zeta^{\mathfrak{m}}(2a+1, 2b+1) = \left(-\delta_{ka} + \binom{2k}{2a} - \binom{2k}{2b}\right) \zeta^{\mathfrak{L}}(2k+1) \otimes \zeta^{\mathfrak{m}}(\mathrm{wt} - 2k + 1)$$

where the binomial coefficient $\binom{x}{y}$ is 0 if $y > x$: there are no ways to choose more than x items from x items.

Consider a linear combination of all the odd-odd DVZs of fixed weight. For illustrative purposes, I'll take weight 12. Then we have:

$$a\zeta(1, 11) + b\zeta(3, 9) + c\zeta(5, 7) + d\zeta(7, 5) + e\zeta(9, 3)$$

For this to be a rational multiple of $\zeta(12)$ we would want all the D_{2k+1} to vanish on this linear combination. That D_9 vanishes, means:

$$\begin{aligned} 0 &= D_9(a\zeta(1, 11) + b\zeta(3, 9) + c\zeta(5, 7) + d\zeta(7, 5) + e\zeta(9, 3)) \\ &= aD_9\zeta(1, 11) + bD_9\zeta(3, 9) = cD_9\zeta(5, 7) + dD_9\zeta(7, 5) + eD_9\zeta(9, 3) \end{aligned}$$

and using the formula for the derivations found above, we get:

$$= (-a - 27b - 42c + 42d + 28e)\zeta^{\mathfrak{L}}(9) \otimes \zeta^{\mathfrak{m}}(5)$$

So that the coefficient $-a - 27b - 42c + 42d + 28e$ should vanish.

A similar calculation for each operator D_{2k+1}, on this linear combination, leads to the following system of equations:

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ -1 & -6 & 0 & 1 & 6 \\ -1 & -15 & -14 & 15 & 15 \\ -1 & -27 & -42 & 42 & 28 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = 0$$

The relations we want lie in the kernel of this matrix M, so we want to find its nullspace. Apply some linear algebra, and one finds a basis for the kernel is given by:

$$v_0 = \begin{pmatrix} 1 \\ \frac{5}{6} \\ \frac{3}{28} \\ 0 \\ 1 \end{pmatrix}, v_1 = \begin{pmatrix} 0 \\ \frac{1}{6} \\ \frac{25}{28} \\ 1 \\ 0 \end{pmatrix}$$

We then reinterpret these vectors as non-trivial relations among the $\zeta(\text{odd}, \text{odd})$, to find the following linear combinations are rational multiples of $\zeta(12)$:

$$\zeta(1, 11) + \frac{5}{6}\zeta(3, 9) + \frac{3}{28}\zeta(5, 7) + \zeta(9, 3) \text{ and}$$
$$\frac{1}{6}\zeta(3, 9) + \frac{25}{28}\zeta(5, 7) + \zeta(7, 5)$$

Scaling up in the second line we immediately recover the non-explicit version of the Gangl-Kaneko-Zagier identity for weight 12 DZVs:

$$28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) \in \zeta(12)\mathbb{Q}$$

Curiously this matrix appears (nearly) as a submatrix of the matrix $Q_k^{(1)}$ from [31], the entries of which come from part of the coefficients of the Fourier series for the double Eisenstein series:

$$G_{ij}(\tau) = \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbb{Z}+\tau\mathbb{Z} \\ \mathbf{m}\succ\mathbf{n}\succ 0}} \frac{1}{\mathbf{m}^i\mathbf{n}^j}$$

The matrix $(Q_k^{(1)})^\top$ does not include the column of -1's corresponding to $\zeta(1,11)$, but the other columns do correspond to the action of the operators D_{2k+1} on the $\zeta(\text{even}, \text{even})$. One knows that the sum of all DZVs of weight N is just $\zeta(N)$, so dropping $\zeta(1,11)$ is no loss. Then finding the nullspace of $(Q_k^{(1)})^\top$ has the effect of finding all linear relations between DZVs, rather than just picking out this 'exceptional' relation.

Facts about the rank of this matrix M, or rather its $(Q_k^{(1)})^\top$ incarnation, and hence results about the number independent relations between DZVs, follow from the theory of periods of $\mathrm{SL}_2(\mathbb{Z})$. But why these exact coefficients should appear from two completely different sources is unclear.

4 Conclusion

In this report we have introduced the polylogarithm and multiple polylogarithm functions. We have explored some properties of these functions, their special values and the functional equations they satisfy. We have seen the fundamental role polylogarithms play in connection with the Dedekind zeta function: Zagier's conjecture proposes a relation between the value of the Dedekind zeta function at n and the order n polylogarithm.

We have looked at the properties of iterated integrals, and how they may be used to encode values of the multiple polylogarithm. Lifting to the motivic level, we see how we gain new algebraic structures, and eliminate transcendence problems. We have introduced the polygon algebra, which, through its connection to iterated integrals and algebraic cycles, captures many of the combinatorial properties of multiple polylogarithms. I have shown how the polygon algebra admits other algebraic structures, such as an operadic composition and a VV-differential, and highlighted the similarities these structures share with other objects. The operadic composition of R-deco polygons looks very much like the composition in the mosaic operad connected with tessellations of moduli spaces. The collapsing of VV-arrows in the VV-differential is reminiscent of terms in the coproduct on dissection diagrams and dissection polylogarithms. Lastly I exhibited suggestions for other avenues of exploration, relating polygons to moduli spaces and maximally degenerate curves.

Then we turned to certain special values of the multiple polylogarithm, the multiple zeta values. We have seen that a great deal of the interest surrounding MZVs stems from the way very simple, basic questions about their transcendence properties and algebraic structure are exceptionally difficult to answer, except conjecturally. Questions even about the irrationality of $\zeta(\text{odd})$ remain almost completely unanswered. We have explained how the recent work of Brown has provided new tools to answer questions about MZVs. With it he re-proves a bound on the dimension of the space of MZVs, and obtains a proof that the Hoffman elements $\zeta(2\text{'s and }3\text{'s})$ span the space of MZVs.

We have then used some of the combinatorial tools provided by Brown's work to investigate relations between MZVs. I have used the infinitesimal coproduct operators D_r, which form the basis for Brown's motivic MZV decomposition algorithm, to re-establish the non-explicit form of the Gangl-Kaneko-Zagier identities. During this computation I have noticed how the coproduct structure on DZVs curiously recovers certain coefficients which arise in the Fourier series of the double Eisenstein series. I have also used this coproduct to prove a certain MZV, conjectured by Borwein, Bradley, and Broadhurst, is indeed a non-explicit rational multiple of a power of π, and generalised this proof to a symmetric insertion result, analogous to the cyclic insertion conjecture of Borwein et al.

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