Lecture 10

Last time: mcg(F), Dehn-Lickorish thm,
Dehn-Lickorish-Wallace thm
(ex. of Dehn surgery presentation)

Today: - Poincaré homology sphere
- (towards) prime decomp. thm

Plan for rest of semester:

- decomposing 3-mfds along spheres & tori
- Last lecture (Jan 29): - Outlook
  - Question session

§4.3 Poincaré homology sphere

First version of Poincaré conjecture (~1900): (✱)

Let M^3 be a closed, conn. 3-mfd.

If M is an (integer) homology sphere,

i.e. $H_*(M;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n=0,3 \\ 0, & \text{otherwise}, \end{cases}$

then $M \cong_{C^0} S^3$.

Rem: M^3 (cld, conn, or.) is a homology sphere

$\overset{\text{Exer 2,}}{\underset{\text{Sheet 3}}{\Longleftrightarrow}}$ $H_1(M) \cong \pi_1(M)_{ab} \cong \{1\}$.

In that case $\pi_1(M)$ is called a perfect gp.

Rem: The equivalent statement to (✱) for 1- & 2-mfds (cld, conn.) is true.

Thm (Poincaré's counterexample, 1904):
(to this first version of his conj.

There exists a 3-mfd P, now called Poincaré homology sphere, s.t. P is a homology sphere, but

$\pi_1(P) = \langle a, b \mid a^3 = b^5 = (ab)^2 = 1 \rangle =: I^*$ w/

$|\pi_1(P)| = 120$, so $\pi_1(P) \not\cong \{1\}$ & $P \not\cong_{C^0} S^3$.

I^* is the so-called binary icosahedral gp.

⇝ Updated version of Poincaré conj. (thm by Perelman 2003):

M^3 s.c., conn., cld 3-mfd $\Rightarrow M \cong_{C^0} S^3$.

simply connected
$\Rightarrow \pi_1(M) \cong \{1\}$

See also the Notes for Lecture 2 for more on the Poincaré conjecture.

⚠ The first version of these notes contained a typo, the generalized Poinc. Conj. should be phrased as

"M^n closed top/smth homotopy sphere (i.e. $M \simeq S^n$)
$\Rightarrow M \cong_{C^0/C^\infty} S^n$."

There are many equivalent definitions of P.

Some of them (& the equivalences) are described e.g. in

Kirby, R. C.; Scharlemann, M. G.
"Eight faces of the Poincaré homology 3-sphere"
Geometric topology, Proc. Conf., Athens/Ga. 1977, 113-146 (1979).

See also http://www.map.mpim-bonn.mpg.de/Poincar%C3%A9%27s_homology_sphere (Manifold Atlas Project).

Poincaré's original def'n: Heegaard diagram of genus 2

[See also Rolfsen 9.D.1]

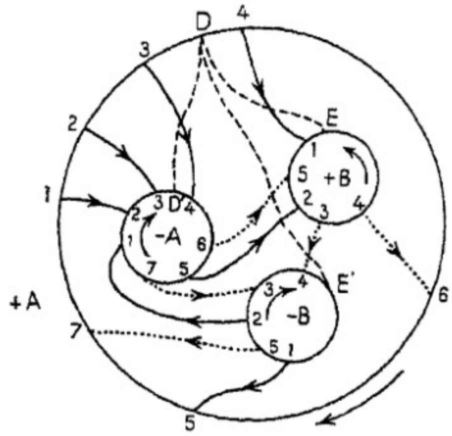


Fig. 1.

Source: H. Poincaré, "Cinquième complément à l'analysis situs" Palermo Rend. 18 (1904), 45-110.

PDF available on Andrew Ranicki's website:
https://www.maths.ed.ac.uk/~v1ranick/papers/poincare.pdf ⇝ p. 494

Description due to Dehn (~1907):

$P =$  $^{+1}$

RHT

(right handed trefoil)

Here, $+1 = \frac{1}{1}$ means $\frac{1}{1}$-surgery along RHT, i.e.

$P \cong S^3_{RHT}(\frac{1}{1}) = RHT_{+1}$ ← just different notations

$= S^3 \setminus \mathrm{Int}(\nu RHT) \cup_\varphi S^1 \times D^2$

where $\varphi : \{1\} \times \partial D^2 \subset S^1 \times D^2$

$\mapsto 1 \cdot \mu_{RHT} + 1 \cdot \lambda_{RHT}$

See Rolfsen 9.D.7 for a proof of the equivalence of these two descriptions of P.

We also have  $^{-1} \cong_{C^\infty} P.$

LHT

(left-handed trefoil)

Pf sketch of Thm: It is not too hard to find $\pi_1(P)$ from the above descriptions.

- From a Heegaard diagram $\overset{\text{Exer 3}}{\underset{\text{Sheet 3}}{\rightsquigarrow}}$ $\pi_1(P)$

- From a surgery presentation $S^3_K(\frac{p}{q})$, e.g.  $+1 = \frac{p}{q}$

  $K = RHT$

  Step I: Find $\pi_1(S^3 \setminus \mathrm{Int}(\nu K))$.

  There's an algorithm to do this, the resulting presentation is called Wirtinger presentation.

  [See e.g. Rolfsen 3.D.2.]

  Step II: Figure out the new relations that appear when gluing $S^1 \times D^2$ to $S^3 \setminus \mathrm{Int}(\nu K)$.

For details on these two steps for P, see Rolfsen 9.D.3. (where P is denoted Q).

Moreover, P is a homology sphere b/c

$$H_1\left(S^3_K\left(\tfrac{p}{q}\right)\right) \cong \langle \mu_K \mid p\,\mu_K = 0 \rangle \cong \mathbb{Z}/p\mathbb{Z}$$

($\forall$ knots K, $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$)

coprime

why? Exer 2 on Sheet 4 shows us that $H_1(S^3 \setminus \mathrm{Int}(\nu K)) \cong \mathbb{Z}$, generated by μ_K.
Gluing $S^1 \times D^2$ to form $S^3_K(\frac{p}{q})$ adds the relation $p\,\mu_K + q\,\lambda_K = 0$, but λ_K bounds a surface (a push-off of a Seifert surface for K; see Exer 6, Sheet 4), i.e. vanishes in homology.

here $p = 1$, so $H_1(P) \cong \{0\}$. $\square$

[**Rem:** More generally, we have

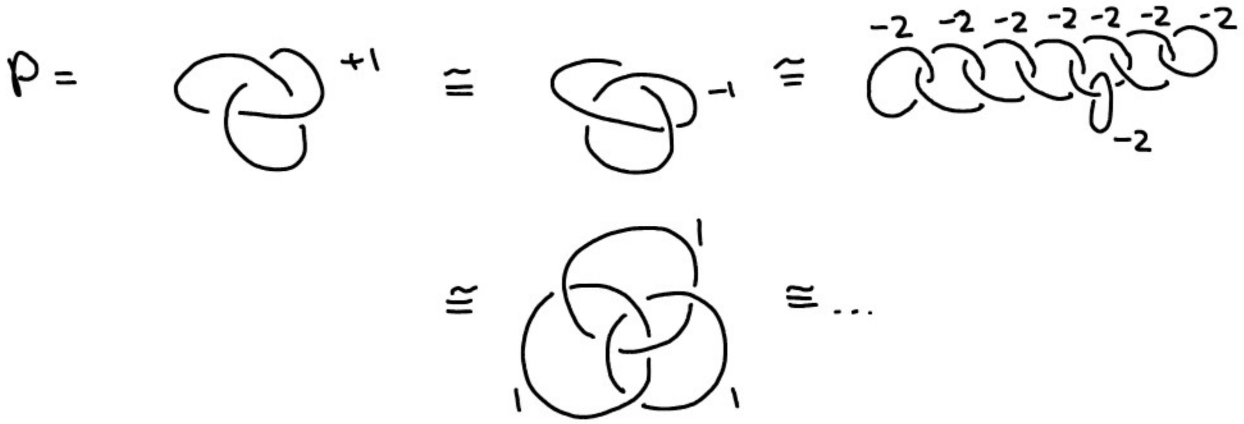
$$H_1(S^3_L(r_1;\dots;r_m);\mathbb{Z}) \cong \langle \mu_i \mid p_i\mu_i + q_i \sum_{j\neq i} \mathrm{lk}(L_i,L_j)\,\mu_j = 0\rangle$$

where $L = L_1 \cup \dots \cup L_m$ w/ meridians $\mu_i = \mu_{L_i}$, $i = 1,\dots,m$.

See e.g. Gompf-Stipsicz, 5.3.11.]

Other descriptions of P:

Just using Dehn surgery presentations and Thm 2 from Lecture 8 (i.e. Rolfsen twists & addition/deletion of link components w/ surg. coeff. $\infty = \frac{1}{0}$), one can find infinitely many Dehn surgery presentations for P, e.g.



[See e.g. §18 in Prasolov-Sossinsky, in part. 18.3, 18.7 & 18.10.]

Rem:
All these integer surgery descriptions (where all surg. coeff. are $\in \mathbb{Z}$) also give rise to descriptions of P as boundary of a smooth 4-manifold (by "Kirby calculus").

Let $f(x,y,z) = x^5 + y^3 + z^2 \in \mathbb{C}[x,y,z]$,

$V(f) := f^{-1}(0) = \{ (x,y,z) \in \mathbb{C}^3 \mid f(x,y,z) = 0 \} \subseteq \mathbb{C}^3$

(complex variety w/ singularity at origin $x=y=z=0$).

Define $L(f) := V(f) \pitchfork S^5_{(\varepsilon)} \subseteq \mathbb{C}^3$. Then $L(f) \cong_{C^\infty} P$.

(called <u>link</u> of the singularity)

(small) sphere around origin

[Dimension count: over $\mathbb{R}$, $L(f)$ has dimension $5+4-6=3$ ($\dim_{\mathbb{R}} V(f) = 4$, $\dim_{\mathbb{R}} \mathbb{C}^3 = 6$).]

[Aside: Similar construction one (complex) dimension lower:

$f(x,y) = x^p - y^q \in \mathbb{C}[x,y]$

$\Rightarrow L(f) := V(f) \cap S^3 \subseteq \mathbb{C}^2 \cong \mathbb{R}^4$, $L(f) = T_{p,q}$

1-dim. submfd of $S^3 \subseteq \mathbb{C}^2$ $\rightsquigarrow$ <u>link</u>

torus knot (Sheet 4, Exer 1)]

$P \cong$ dodecahedron $/\sim$

regular solid w/ 12 faces, 30 edges, 20 vertices

identify opposite faces of the dodecahedron via twist by $\frac{2\pi}{10} = 36°$.

$P \cong S^3 / I^*$ where I^* binary icosahedral gp

$\downarrow 2:1$

$I = \mathrm{Isom}$ (icosahedron)

symmetry gp of regular icosahedron

$\updownarrow$ dual polyhedron

dodecahedron

($\exists$ a short exact sequence $1 \to \mathbb{Z}/2\mathbb{Z} \to I^* \to I \to 1$.)

§ 5 Decomposing 3-manifolds along spheres & tori

§ 5.1 Connected sum & prime decomposition

Main sources for this section:

A. Hatcher, Notes on Basic 3-Manifold Topology, in particular Chapter 1.1 "Prime Decomposition"
B. Martelli, An Introduction to Geometric Topology, in particular Sections 1.1.12, 1.1.13 and 9.2
J. Schultens, Introduction to 3-Manifolds, in particular Sections 1.6, 5.5 and 6.3
J. Hempel, 3-Manifolds, in particular Chapter 3

Similar to the connected sum for surfaces, we define the conn. sum for 3-manifolds.
↳ see Lecture 1

not just orientable

Def: Let M_1^3, M_2^3 be connected, oriented 3-manifolds.

Let $D_i^3 \subset \mathring{M}_i^3$, $i=1,2$, be embedded 3-balls & let

$\varphi: D_2^3 \to D_1^3$ be an or.-rev. diffeom.

Then
$$M_1 \# M_2 := M_1 \setminus \operatorname{Int}(D_1^3) \cup_{\varphi|_{\partial D_2^3}} M_2 \setminus \operatorname{Int}(D_2^3)$$

is called the connected sum of M_1 & M_2.

Lemma: (1) This is a well defined (smooth) 3-mfd (up to diffeo).

[See e.g. Martelli, Prop. 1.1.16]

(2) Connected sum is commutative and associative.

We write e.g. $\#_m M := \underbrace{M \# \ldots \# M}_{m \text{ times}}$.

(3) $M \# S^3 \cong_{C^\infty} M \quad \forall M^3$. (Pf at end of these notes)

Regarding the pf of (1):
- similar to well def'dness of ↯
- Key words:
  - smoothen corners
  - Disk Thm (Palais)
  - isotopic gluing maps give diffeo mfds

Regarding (2):
- Commutativity by def'n of #
- Assoc.: Can assume that $\partial D_{j_i}^3 \subset M_i$, $i=1,2$, are disjoint

Rem: There's a well def'd # for M_1^n, M_2^n w/ (2) for $n \neq 2,3$, too, but we have to fix a canonical way of gluing (for higher n).

[Reason: Milnor's exotic 7-spheres.]

Rem: For orientable, but not oriented 3-mfds, there exist examples where $M_1 \# M_2 \not\cong_{C^\infty} M_1 \# \underset{\substack{\uparrow \\ M_2 \text{ w/ reversed orientation}}}{-M_2}$ (see e.g. Hempel, Ex. 3.22, p. 36).

Exer: 1) How do π_1 & H_1 behave under connected sum?

Show that
- $\pi_1(M_1 \# M_2) \cong \pi_1(M_1) * \pi_1(M_2)$ (using the Seifert-van Kampen thm)
- $H_1(M_1 \# M_2) \cong H_1(M_1) \oplus H_1(M_2)$.

2) Given
- Heegaard diagrams or
- Dehn surgery presentations

of M_1^3 & M_2^3,

find a Heegaard diagram or Dehn surg. pres. for $M_1 \# M_2$, resp.

Throughout the rest of this section (if not stated otherwise), let M^3 be an oriented, conn. (smooth) 3-mfd & suppose that all considered submanifolds of M are <u>smoothly embedded</u>. (as usual in this course, but let me stress it again).

Def: Let M^3 be an or., conn. 3-mfd & let $S^2 \subset \mathring{M}$ be a cld, or., smoothly emb. 2-dim. submfd.

__Cutting M along S__ means to consider

$$M' := M|S := M \setminus \underbrace{\mathring{\nu} S}$$ "M cut along S"

open tubular/regular nbhd of S in M (i.e. nbhd of S which is diffeo to $S \times (-1,1)$)

[see e.g. Def. 3.1.12, Def. 3.2.4 & Thm 3.1.14 in Schultens]

$\rightsquigarrow$ M' is a 3-mfd w/ two new boundary components $S \times \{-1\}$ & $S \times \{1\}$, both $\cong_{C^\infty} S$.

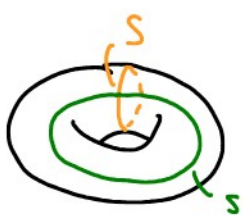
Def: M, S as above, suppose S is __connected__. S is called

- __separating__ if $M' = M|S$ has two components
- & __non-separating__ otherwise. (Recall: M __connected__)

Rem: S is non-sep. $\iff$ $\exists$ __s.c.c.__ (simple closed curve) in M that intersects S transversely in __one__ point.

Ex $(S^1 \subset M^2)$ $\rightsquigarrow$ one dimension lower we can make the same definitions

$M' = M|S$



__non__-sep.

s.c.c. as in Rem. above

separating

Back to #:

Let M^3 be or, conn. & $S^2 \subset \mathring{M}$ a <u>separating</u> sphere,

(i.e. $\exists\ S \xrightarrow[\cong_{C^\infty}]{\psi} \underset{\partial D^3}{\overset{\shortparallel}{S^2}}$).

Call the two components of $M \setminus S$ M_1' & M_2'.

Glue balls D_i^3 to M_i^3, $i=1,2$, via ψ to obtain M_1 & M_2.

$\Rightarrow M = M_1 \# M_2$.   [Indeed, M_1 & M_2 are <u>uniquely</u> det'd by M_1' & M_2' (up to diffeo); cf. Step 3 in Lecture 5.]

So we can go back and forth between the summands M_1 and M_2 & their connected sum $M = M_1 \# M_2$ if we find the corresponding separating sphere.

<u>Pf of (3) of Lemma:</u>

(NTS $M^3 \cong_{C^\infty} M \# S^3$)

Let M^3 be or., conn. Choose a <u>separating</u> S^2 in M which <u>bounds a ball</u> (in M). (We can do so b/c M^3 locally looks like $\mathbb{R}^3$.)

$\Rightarrow M \setminus S$ has two components $\underset{D^3}{\overset{C^\infty\shortparallel\wr}{M_1'}}$ & $\underset{M \setminus D^3}{\overset{C^\infty\shortparallel\wr}{M_2'}}$.

$$\Rightarrow M_1 := M_1' \underset{\substack{\uparrow \\ \text{as above}}}{\cup_\psi} D^3 \cong D^3 \cup_\psi D^3 \underset{\substack{\uparrow \\ \text{Exer 4} \\ \text{Sheet 2}}}{\cong} S^3$$

$$M_2 = M_2' \cup_\psi D^3 \cong M \setminus D^3 \cup_\psi D^3 \cong M.$$

$$\Rightarrow M = M \# S^3.$$

This shows that we can always "split off" a summand S^3.

The other way round, given $M_1 \cong M$ & $M_2 \cong S^3$,

we also obtain $M \# S^3 \cong M_1 \# M_2 \cong M$ (using Palais' Disk Thm cf. Lect. 4). $\square$

Def: M^3 (conn., or.) is prime if every decomposition of M as connected sum $M \cong M_1 \# M_2$ is trivial, i.e. $M_1 \cong S^3$ or $M_2 \cong S^3$.

Otherwise such a conn. sum decomp. is called non-trivial.

Thm (Prime decomp thm) → more details in Lect. 11

Every cpct, conn., or. 3-mfd M (possibly w/ $\partial M \neq \emptyset$) decomposes into prime manifolds, i.e. $\exists$ $P_1, \ldots, P_r$ prime, or., conn. s.t. $M \cong_{C^\infty} P_1 \# \ldots \# P_r$.