Lecture 11

Last time: connected sum #,

M^3 prime $:\Leftrightarrow$ $(M \cong M_1 \# M_2 \Rightarrow M_1 \cong S^3$ or $M_2 \cong S^3)$

Today:
- pf of existence in prime decomp. thm
- examples of prime mfds $\rightsquigarrow$ irreducibility

Thm 1 (Prime decomp thm – closed case; Existence + Uniqueness):

Every closed, conn., or. 3-mfd M decomposes into prime manifolds, i.e. $\exists$ $P_1, \dots, P_r$ prime, or., conn. 3-mfds s.t. $M \cong_{C^\infty} P_1 \# \dots \# P_r$.

This decomposition is unique up to insertion / deletion of S^3-summands & reordering of the summands, in particular unique up to or. pres. diffeom. of the P_i.

The first proof of the existence is due to Kneser (1929) using triangulations & "normal" surfaces to bound the nr. of separating spheres we can find in M.

See e.g. most text books that cover the prime decomp. thm,

e.g. Schultens, Thm 5.5.1 & Thm 5.5.3
Hatcher, Thm 1.5
Hempel, Thm 3.15 (existence) & Thm 3.21 (uniqueness)

Uniqueness is due to Milnor (1962).

We will sketch a pf of the existence (in the closed case) following W. H. Jaco, Lectures on Three-Manifold Topology, AMS, 1980 (Thm II. 4) & using the Heegaard genus.

Let M^3 be a cld, or., conn. 3-mfd.

Recall from Lectures 6 & 7 (§3.3) that

Heegaard genus $(M) := \min\{g(\Sigma) \mid M = H_1 \cup_\Sigma H_2$ Heegaard splitting of $M\}$

and

(*)
- $g(M) = 0 \overset{\text{Lect 6}}{\Longleftrightarrow} M \cong_{C^\circ} S^3$
- $g(M) = 1 \overset{\text{Lect 6\&7}}{\Longleftrightarrow} M \cong L_{p,q}$ for $p \neq 1$ (i.e. $\not\cong_{C^\circ} S^3$) including $L_{0,1} = S^1 \times S^2$.

[i.e. $M \cong L_{p,q}$ for some $1 < q < p$ or $(p,q) = (0,1)$.]

To show the existence of a prime decomp., we will need:

Thm 2: The lens spaces $L_{p,q}$, including $S^3, S^1 \times S^2$, are all prime.

Prop 1: ("Heegaard genus is additive")

Let M_1, M_2 be cld, or., conn. 3-mfds.
Then $g(M_1 \# M_2) = g(M_1) + g(M_2)$.

We'll say something about the pfs / pf ideas of Thm 2 & Prop. 1 later. For now, let us turn to the pf of Thm 1.

Pf of Thm 1 (existence):

Let M be a cld, or, conn. 3-mfd. We prove the existence of a prime decomposition for M by induction on its Heegaard genus $g(M) =: n$.

First, suppose that $g(M) \in \{0,1\}$. $\underset{\&\ (*)}{\overset{\text{Thm 2}}{\Longrightarrow}}$ M is prime.

Now, let $n \geq 2$.

IH: Every cld, or., conn. 3-mfd w/ $g(M) \leq n-1$ admits a prime decomposition

IS: $(g(M)=n)$

- If M is prime, we're done.

- If not, then $M \cong_{C^\infty} M_1 \# M_2$ for $M_i \not\cong S^3$, $i=1,2$ (by def'n).

Prop 1 $\Rightarrow$ $g(M) = \underbrace{g(M_1)}_{\neq 0} + \underbrace{g(M_2)}_{\neq 0}$, so $0 < g(M_i) \leq n-1$, $i=1,2$.

by (*)

By IH, both M_1 & M_2 have a prime decomp.

which gives a prime decomp. of $M \cong M_1 \# M_2$. □

Rem: Indeed, the above pf shows that

- # summands of prime decomp of $M = r \leq g(M)$.

Moreover, • $g(M) = \sum_{i=1}^{r} g(P_i)$ if $M \cong P_1 \# \ldots \# P_r$.

Before giving examples of prime manifolds, let us introduce another very related terminology.

Def: A (cld, or., conn.) 3-mfd M^3 is called <u>irreducible</u> if every smoothly emb. sphere, $S^2 \cong_{C^\infty} S \subseteq M$, bounds a ball in M, i.e. $\exists\, D^3 \cong_{C^\infty} B \subseteq M$ w/ $\partial B = S$.

Lem: Irreducible manifolds are prime.

Pf: Suppose M is irred. and $M = M_1 \#_S M_2$ for some $S \cong S^2$.

Cut M along S $\Rightarrow$ $M \setminus S = M_1' \cup M_2'$. Since M is irreducible, S bounds a ball in M, so $M_1' \cong D^3$ or $M_2' \cong D^3$, WLOG $M_1' \cong D^3$.

$\Rightarrow M_1 \cong M_1' \cup_S D^3 \cong D^3 \cup_S D^3 \cong S^3$. □

Sheet 2, Exer 4

Rem: Similarly, one can show that M^3 is prime if and only if every <u>separating</u> 2-sphere $S \subset M$ bounds a ball.

<u>Thm</u> (Alexander 1924) $\mathbb{R}^3$ is irreducible, i.e. every <u>smoothly</u> embedded $S^2 \subset \mathbb{R}^3$ bounds a ball.

Original pf in the PL category, smooth pf due to Brown; see e.g. Hatcher, Thm 1.1
Schultens, Thm 3.2.5
Martelli, Thm 9.2.10.

<u>Rough pf sketch:</u> Let $S \subseteq \mathbb{R}^3$ be a smoothly emb. sphere. We can suppose that the <u>height</u> <u>function</u> $h: S \to \mathbb{R}$ (given by the z-coordinate) is a <u>Morse fct</u> w/ all critical pts at distinct levels. At a noncritical value a_i of h, we have $h^{-1}(a_i) = \bigsqcup_{k=1}^{r_i} S^1$. By the <u>2D-version of the thm</u> we're proving, each of these circles bounds a disk in the plane $z = a_i$. Consider an <u>innermost</u> such circle C & use the disk it bounds to <u>surger</u> S along C ($\rightsquigarrow S' = S|C$). One can show that after iterating this proces, the surfaces obtained are all spheres of certain "basic" types which all bound balls. Finally, one can inductively show that reversing this cutting process we get back the sphere S which bounds a ball. (At each "backward" step 2 spheres bounding balls are replaced by one sphere where the interiors of these balls are either disjoint or contained in each other.)

<u>2D-version:</u> Any smoothly emb. S^1 in $\mathbb{R}^2$ bounds an emb. disk. (Smooth Jordan Curve Theorem).

Cor: S^3 is irreducible, in fact every smoothly emb. $S^2 \subset S^3$ bounds a ball on both sides.

Pf: Let $S \cong_{C^\infty} S^2 \subset S^3$ be a smoothly emb. sphere in $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

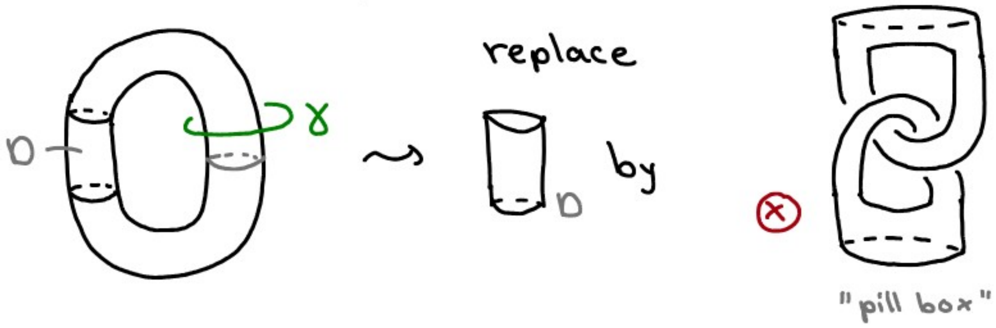
Since S is compact, after an isotopy, S misses ∞, so $S \subseteq \mathbb{R}^3 \subseteq S^3$. By Alexander's Thm, S bds a ball in $\mathbb{R}^3$, thus in S^3.

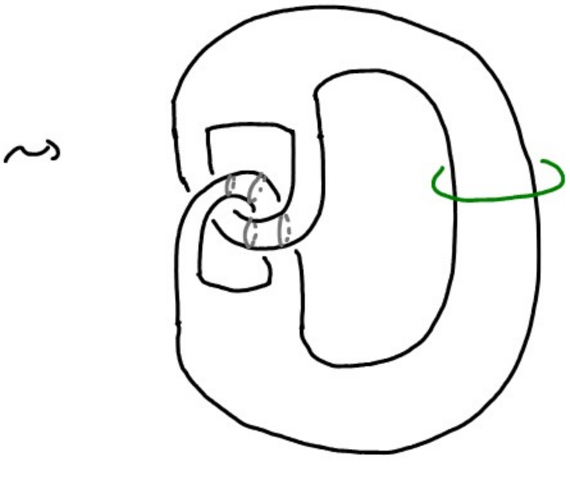
Exer: finish the pf. ($\rightsquigarrow$ ball on both sides) $\square$

Cor: S^3 is prime.

Rem: $\exists$ topological embedding $S^2 \cong_{C^0} S \xhookrightarrow{top} S^3$ s.t. $S^3 \setminus S \not\cong_{C^0} D^3$, e.g. the Alexander horned sphere. (AHS)

Construction of AHS:





Iterate this construction (in each step replace two solid cylinders by ⊗) inf. many times

$\rightsquigarrow \exists\ B$ top. ball $(\cong_{C^0} D^3)$ "Alexander horned ball" w/ $\partial B =: AHS$.

One can show that $S^3 \setminus B$ is not s.c. (b/c γ is not contractible in $S^3 \setminus B$).

"Alexander gored ball"

So AHS does not bound a ball on both sides in S^3. (*)

A nice visualization of AHS can be found e.g. here:

- https://en.wikipedia.org/wiki/Alexander_horned_sphere
- https://www.youtube.com/watch?v=d1Vjsm9pQlc

For a bit more formal treatment of the construction & details on the pf of (*) see e.g. p. 47-51 in

R. J. Daverman and G. Venema, Embeddings in Manifolds, AMS, 2009.

Lem / Ex: • $S^1 \times S^2$ is prime, but not irreducible.

• Indeed, if M^3 is cld, or., conn. and prime, then M is irreducible or $M \cong S^1 \times S^2$.

Pf: • $S^1 \times S^2$ is not irreducible b/c $S := \{pt\} \times S^2$ doesn't bound an emb. ball [e.g. b/c it generates $H_2(S^1 \times S^2) \cong \mathbb{Z}$].

• Cl: $S^1 \times S^2$ is prime.

Pf: We'll use the above remark / fact that M^3 is prime iff every separating 2-sphere $S \subset M$ bounds a ball.

Let $S \subset S^1 \times S^2$ be a separating 2-sphere. NTS: S bounds a ball.

$\Rightarrow$ $S^1 \times S^2 \setminus S = M_1' \dot\cup M_2'$ has two components M_1' and M_2' which are both cpct 3-mfds w/ $\partial M_i' \cong S \cong S^2$, $i = 1, 2$.

We have $\pi_1(S^1 \times S^2) \underset{\text{SvK}}{\overset{\text{Exer}}{\cong}} \pi_1(M_1') * \pi_1(M_2')$

$\cong \pi_1(S^1) \times \pi_1(S^2) \cong \mathbb{Z}$

$\Rightarrow$ WLOG $\pi_1(M_1') \cong \{1\}$, i.e. M_1' is s.c.

The universal cover of $S^1 \times S^2$ is $\widetilde{S^1 \times S^2} = \mathbb{R} \times S^2 \cong \mathbb{R}^3 \setminus \{0\}$ and M_1' lifts to a diffeomorphic copy $\widetilde{M_1'}$ of itself in $\widetilde{S^1 \times S^2}$.

Now, $\partial \widetilde{M_1'} \cong S^2$ bounds a ball in $\widetilde{S^1 \times S^2} \cong \mathbb{R}^3 \setminus \{0\} \subseteq \mathbb{R}^3$ by Alexander's Thm, but $\partial \widetilde{M_1'}$ also bounds $\widetilde{M_1'}$ in $\mathbb{R}^3$.

$\Rightarrow$ $M_1' \cong \widetilde{M_1'} \cong D^3$, so $S \cong \partial M_1'$ bounds a ball as claimed.

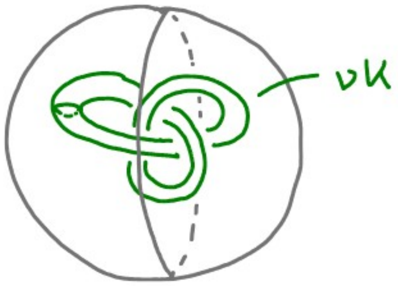
The pf of the last statement can be found e.g. in

- Hatcher, Prop. 1.4
- Martelli, Prop. 9.2.14
- Schultens, Thm 3.3.4. $\square$

Ex: Every cpct 3-mfd $M \subset S^3$ w/ connected boundary is irreducible, e.g.

- Knot exteriors $S^3 \setminus \mathring{\nu} K$ w/ boundary $\partial \nu K \cong S^1 \times S^1$
- handlebodies (which are $\cong_{C^\infty} \natural_g (S^1 \times D^2) \subseteq S^3$).

why? Every $S^2 \subset S^3$ bounds a ball on both sides (see Cor. above). Since ∂M is connected, it is contained in one of these balls, so the other ball is contained in M.



Prop: $p: M \to N$ any covering map of 3-mfds. If M is irreducible, then N also is.

Pf: See e.g.
- Hatcher, Prop. 1.6.
- Martelli, Prop. 9.2.16.

[Rem: The converse of this Prop. is also true provided that N is orientable (i.e. then N irred. $\Rightarrow$ M irred.), but requires more work; see e.g. Thm 3.15 in Hatcher.]

Cor: • $L_{p,q}$ is irreducible, thus also prime for $p \neq 0$ (universal cover is S^3)

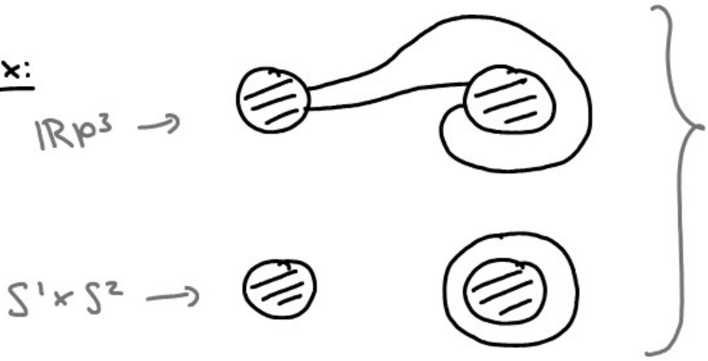
• $S^1 \times \Sigma_g$, $g \geq 1$, is irreducible, e.g. $T^3 = S^1 \times S^1 \times S^1$

(universal cover is • $\mathbb{R} \times \mathbb{R}^2$ $(g=1)$
• $\mathbb{R} \times \mathbb{H}^2$ $(g \geq 2)$).

Pf idea of Prop 1 (Cl: $g(M_1 \# M_2) = g(M_1) + g(M_2)$)

Exer: $g(M_1 \# M_2) \leq g(M_1) + g(M_2)$.

Ex:



Heegaard diagram of $\mathbb{R}P^3 \# S^1 \times S^2$.

The other inequality follows from

Thm (Haken's Lemma): If M cld, or., conn. 3-mfd and M is reducible (i.e. not irreducible), then each Heegaard splitting $M = H_1 \cup_\Sigma H_2$ is reducible, i.e.

$\exists$ 2-sphere $S \subset M$ s.t. $S \cap \Sigma$ is an essential s.c.c. α,

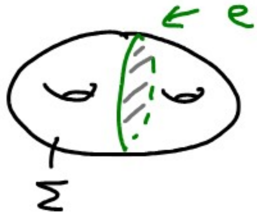
i.e. α is non-separating or separating, but no component of $\Sigma \setminus \alpha$ is a disk / annulus.

typo on board!

[Such a sphere $S \subset M$ gives rise to a decomp. $M = M_1 \# M_2$ and ultimately a Heegaard splitting of both summands M_1 & M_2.]

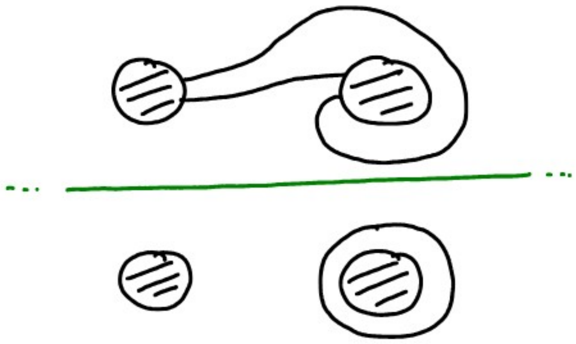
For the pf of Haken's Lemma, see e.g. Schultens, Thm 6.3.5.

Ex:



$\leftarrow$ essential $S \cap \Sigma$ in genus 2 Heegaard splitting of S^3.

Ex:



$\leftarrow$ ess. $S \cap \Sigma$ in Heegaard splitting of $\mathbb{R}P^3 \# S^1 \times S^2$