Lecture 12

! Unfortunately, I noticed some typos on the blackboard in this lecture. I will comment on those in this colour (purple) in the notes below.

§5.2 Torus decomposition

Motto: Cutting 3-mfds along spheres was successful ($\rightsquigarrow$ prime decomp.), let's cut along other surfaces (the next simple closed ones are tori).

To be able to state the torus decomposition theorem due to Jaco, Shalen and Johannson, we will need a few definitions.

First, let's define Seifert fibered spaces.

Seifert fibered spaces

(first studied & classified by Seifert in 1933)

Def: A fibered solid torus of type (ℓ, m) is

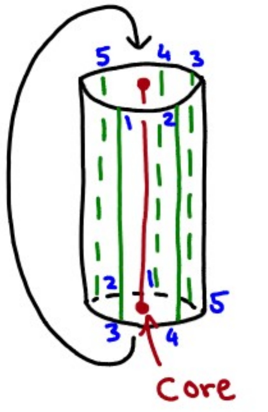
$$T := D^2 \times [-1,1] \Big/ (re^{2\pi i\theta}, 1) \sim (re^{2\pi i(\theta + \frac{m}{\ell})}, -1) \cong D^2 \times S^1$$

where $m \in \mathbb{N} \cup \{0\}$, $\ell \in \mathbb{N}$ (coprime)

w/ fibers $\{pt\} \times [-1,1]/\sim$ & core $\{0\} \times [-1,1]/\sim$

e.g. $\ell = 5$, $m = 2$

Identify disks $D^2 \times \{-1\}$ & $D^2 \times \{1\}$ after a "twist", more precisely a $2\pi \frac{m}{\ell}$-rotation.



$\rightsquigarrow$ T is a solid torus w/ a fibration / partition into disjoint circles:

$\{0\} \times [-1,1]/\sim$ is one fiber, the core, of T.

The other fibers of T are made from ℓ segments $\{x\} \times [-1,1]$ (which wind around the core ℓ times)

[ℓ =: multiplicity of the central fibre / core of T
(= # intersections of central fiber w/ small disk transverse to it)]

If $\ell > 1$, T is called an exceptionally fibered solid torus whose core is an exceptional fiber (or multiple / singular fiber) & all other fibers are regular.

If $\ell = 1$, T is called regularly fibered.

[In the case that $\ell = 1$, T is diffeo to the product fibration $D^2 \times S^1$]

[See e.g. Schultens, Def. 3.7.3 – 3.7.4 or Hatcher, p. 13-14.]

A Seifert fibered space (SFS) is an or., cld M^3 which is the union of pairwise disjoint s.c.c. called fibers s.t. each fiber in M has a tubular neighborhood that is homeomorphic to a fibered solid torus via a fiber-preserving homeo. (fibers $\mapsto$ fibers)

[See Def. 3.7.3, Schultens for cpct case. $\rightsquigarrow$ $\partial M = \sqcup (S^1 \times S^1)$
$\rightsquigarrow$ Exceptional fibers isolated & in int(M).]

Let $B := M/_{\text{identify each fiber to a pt}}$ (quotient space), called base space.

<u>Fact:</u> B is (topologically) a compact surface and the quotient map $p: M \to B$ is a fiber bundle on the complement of the exceptional fibers.

[• <u>Ex:</u> $M \cong_{C^0}$ fibered solid torus w/ $\ell = 1$ $\implies$ $B \cong D^2$
& $p: M \to B$ fiber bundle in usual sense.

• For a fibered solid torus T w/ $\ell > 1$, we still obtain $T/_{\sim} \cong D^2$, but now this disk looks like a "cone" [quotient of D^2 by $\mathbb{Z}/\ell$-act.] (rotation by $2\pi \frac{m}{\ell}$)

(identify fibers to pts)

and $T \to T/_{\sim}$ is really an "orbifold bundle". ("circle bundle over orbifold B")

See • Martelli, Prop. 10.3.8
• Schultens, Rem. 3.7.11
• P. Scott, The Geometries of 3-Manifolds, 1983, p. 430
https://doi.org/10.1112/blms/15.5.401

See Section 6.2.4, Martelli, for more on orbifolds.]

fiber   base space

Ex: Any S^1-bundle over a surface, e.g. $\Sigma_g \times S^1$,

(trivial S^1-bundle over the cld surf. of genus g)

is a Seifert fibered space (w/ no exceptional fibers).

[Follows from def'n of such fiber bundles.]

Ex: The lens spaces $L_{p,q}$ are SFS. (See Schultens, p. 89 w/ different conventions)

Recall that $L_{p,q} = H_1 \cup_\varphi H_2$ where both $H_i \cong S^1 \times D^2$, $i = 1, 2$, and φ corresponds to $\varphi_* \in Aut(\pi_1(T^2)) \cong GL_2(\mathbb{Z})$ and thus a matrix $\begin{pmatrix} q & p \\ r & s \end{pmatrix}$ (see Lect. 7).

In particular φ_* maps a basis (μ_1, λ_1) of $\pi_1(\partial H_2)$ to $(q\mu_1 + p\lambda_1, r\mu_1 + s\lambda_1)$

and μ_2 gets identified w/ $q\mu_1 + p\lambda_1$, a curve that meets μ_1 in p pts & λ_1 in q pts.

Take a fibration on H_1 & H_2 that matches on $\partial H_1 \cong \partial H_2$.

One way to do this: Take s.c.c. on $\partial H_1 \cong \partial H_2$ that intersects μ_2 once, e.g. a s.c.c. that intersects μ_1 in $r \neq 0$ & λ_1 in s (equivalently or.) pts. Give H_i the structure of a fibered solid torus s.t. this curve is a regular fiber. $\Rightarrow L_{p,q}$ SFS.

Ex: $S^3 \cong L_{1,q}$ is a SFS w/ except. fibers 

(Hopf fibration)

Ex: Exteriors of torus knots $T_{p,q}$ are SFS.

Why? $S^3 = H_1 \cup H_2$, $H_i \cong S^1 \times D^2$. Let p, q be coprime and give

• H_1 the structure of a fibered solid torus of type (p,q)

and • H_2 ————— " ————— (q,p).

When we glue ∂H_1 to ∂H_2 to form S^3, the fibers on the boundaries are identified, so S^3 is a SFS.

Now, a regular fiber on ∂H_1 is a torus knot $T_{p,q}$ (check!).

So if we remove a fibered solid torus nbhd of this regular fiber, the result is a SFS homeomorphic to the exterior of a torus knot, i.e. $S^3 \setminus \mathring{\nu} T_{p,q}$.

[See Ex. 8.20 & 8.21 in J. S. Purcell. Hyperbolic Knot Theory. AMS, 2020.]

Rem: There's a classification of Seifert fibered spaces, see e.g.

- Hatcher, Prop. 2.1 & Thm 2.3
- Martelli, Sections 10.3.2–10.3.3.

Incompressible surfaces

References for this section:
- Schultens, Sect. 3.4, 3.5
- Martelli, Sect. 9.3, 9.4
- Hatcher, Sect. 1.2
- Jaco, (beginning of) Chapter III
- Hempel, Chapter 6

Througout, let M^3 be a cpct, or. 3-mfd, possibly w/ $\partial M \neq \emptyset$.

Suppose that $S^2 \subset M$ (← surface) is a proper submfd, i.e. $\partial S = \partial M \cap S$,

and S is cpct, or., conn. $(*)$.

Def: M, S as above. S is called compressible if

(1) $S \cong S^2$ and S bounds a ball in M or

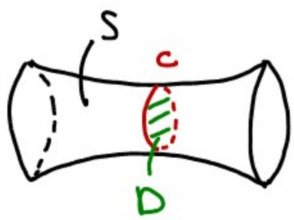
(2) $\exists$ s.c.c. c in S s.t. c bounds a disk D w/ $\mathring{D} \subset M \setminus S$
(simple cld curve)
but c does not bd a disk D' w/ $\mathring{D}' \subset S \setminus c$.

Such a disk D is called a compressing disk for S.

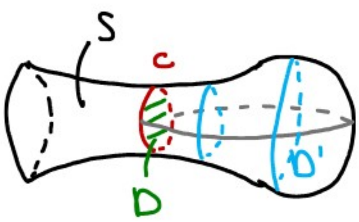
If S is not compressible, then it is incompressible.

[Note that D' as in (2) always exists if $S \cong S^2$. Some authors exclude case (1). We follow Jaco's def'n (see his p.31).]

Ex:

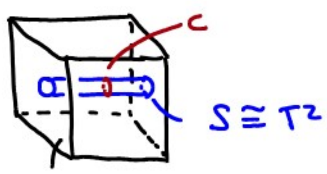


w/ $\mathring{D} \subset M \setminus S$



$S \not\cong S^2$ incompressible if $\forall$ disks $D \subset M$ w/ $\partial D = D \cap S = c$ $\exists$ disk $D' \subset S$ w/ $\partial D' = c$.

Ex



compressible b/c c bds disk in T^3, but not in T^2.

T^3 (quot. of cube)
see Sheet 2, Exer 3a

<u>Def / Construction:</u> Let $D \subset M$ be a compressing disk for $S \subset M$.

Then <u>surgery</u> of S along D $(w/ \partial D = D \cap S)$

is $S' := S \setminus \nu(\mathring{\partial D}) \cup D_1 \cup D_2$

where $D_1 \cong D_2 \cong D$.

← add two parallel copies of D, so S' has <u>no</u> new bdry comp.

(<u>not</u> $S \setminus \partial D$, but $S \setminus \partial D \cup D_1 \cup D_2$!)

! I think I said this correctly in the lecture and drew the correct picture (see below), but didn't write the def'n entirely correct on the board - sorry! :^

<u>Ex:</u>

<u>Prop:</u> (1) • If $S \subset M$ is incompr., then surgery of S along D (S' as above), where $D \subset M$ is s.t. $\partial D = D \cap S$, just splits off an S^2 (and has a diffeom. copy of S as the other piece resulting from the surg.).

(2) • Let $S \subset M$ be as in $(*)$, but not necessarily connected.

(so prop. emb., cpct, or.)

After <u>compressing</u> (surgery along compr. disk) S a finite number of times, we obtain a disjoint union of spheres, disks and incompr. surfaces.

<u>Pf of (2):</u> Use Euler characteristic to show that the process of compressing S as much as we can must stop. See Cor. 9.3.2 / Prop. 9.3.1, Martelli.

There are three important related results in 3-mfd theory that were all proved by the Greek mathematician Papakyriakopoulos in the 1950's:

- Dehn's lemma (thought to be proven by Dehn much earlier)
- the Loop theorem
- the Sphere theorem

The common theme of these theorems is to promote a cont. map from a disk / sphere into M (or $(M, \partial M)$) to an embedding. Especially the latter two thms relate homotopy theoretic properties of a 3-mfd (π_1 & π_2) to more geometric properties. We will only state the last of these 3 theorems.

<u>Sphere Thm</u> (Papakyriakopoulos 1957, Whitehead 1958)

Let M^3 be an or., conn. 3-mfd w/ $\pi_2(M) \neq \{0\}$.

Then there exists an embedding $g: S^2 \hookrightarrow M$ s.t. $[g] \neq 0 \in \pi_2(M)$.

The <u>pf idea</u> is to use a tower of covering spaces.

[Original sources:

C.D. Papakyriakopoulos. ON DEHN'S LEMMA AND THE ASPHERICITY OF KNOTS (1957). Proceedings of the National Academy of Sciences of the United States of America, 43(1), 169–172.

J.H.C. Whitehead. "On 2-spheres in 3-manifolds." Bulletin of the American Mathematical Society 64 (1958): 161-166.

See also Schultens 3.5.4, Hempel 4.3 & 4.11.]

The following lemma is a consequence of the <u>Loop thm.</u>

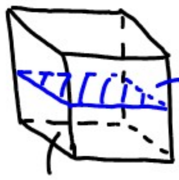
<u>Lem:</u> $S \subset M$ as above, $S \not\cong_{C^\infty} S^2$, is incompressible $\Leftrightarrow$ the map $\pi_1(S) \to \pi_1(M)$ induced by the inclusion is injective.

[Note that $\pi_1(S^2) \cong \{1\} \hookrightarrow \pi_1(M)$ $\forall S \cong S^2 \subset M$, e.g. for compressible S^2 in $M \cong S^3$, so we should exclude the case $S \cong S^2$!]

[<u>Sketch of "$\Leftarrow$":</u> Suppose the map is inj. and D is a compr. disk for S. Then ∂D is a s.c.c. in S which does not bd a disk D' in S. It can be shown that such a s.c.c. cannot be nullhomotopic, i.e. ∂D represents a non-trivial element in $\pi_1(S)$. But it doesn't in $\pi_1(M)$. $\lightning$]

[For the pf of "$\Rightarrow$", see e.g. • Hatcher, Cor. 3.3 or
• Martelli, Thm 9.4.14.]

<u>Ex:</u>



$S \cong T^2$ incompressible (follows from the Lemma)

T^3

<u>Ex:</u> $\mathbb{R}^3$ and S^3 do <u>not</u> have any emb. incompr. surfaces.

(no incompr. S^2 b/c irreducible; no other incompr. surf. by the lemma)
(see Lect. 11)

<u>Lem / Fact:</u> For any emb. torus T in an irred. 3-mfd M, we have one of the following:

- T is incompressible
- T bounds a solid torus
- T is contained in a ball in M.

[<u>Pf:</u> See • (3) on p. 14 in §1.2 in Hatcher or
• Prop. 9.3.6 in Martelli; see also Fig. 9.17 there.]

<u>Ex./ Consequence:</u> Any emb. $T^2 \subset S^3$ bounds a solid torus.
(which can be knotted!)

(the 3rd case can be excluded in S^3: if $T \cong T^2$ is contained in a ball B, then it bounds a solid torus "on the outside" using the complementary ball $S^3 \setminus B$.)

<u>Ex:</u>

knot exterior

$X_K := S^3 \setminus \mathring{\nu} K$, K knot

$T := \partial X_K = \partial \nu K \cong S^1 \times S^1$

T is incompressible in X_K iff $K \underset{\text{iso}}{\ncong}$ unknot.
(& doesn't bound a solid torus in X_K)

[see e.g. Thm 4.B.2 in Rolfsen]

An example for the second case of the above Lem/Fact is easy to find. Here's an example for the third case:

[see e.g. • Rolfsen, p. 106
• Martelli, Fig. 9.17]

<u>Rem:</u> Without making this precise, let me mention that incompr. surfaces in SFS can be classified into "horizontal" and "vertical" ones, see e.g. • Prop. 1.12, Hatcher
• Thm 3.7.16, Schultens.

<u>Lem/Fact:</u> Let $S \subset M$ be incompressible. Then

$$M \text{ is irreducible} \iff M|S \text{ is irreducible.}$$

$M|S$: M cut along S (see Lect. 10)

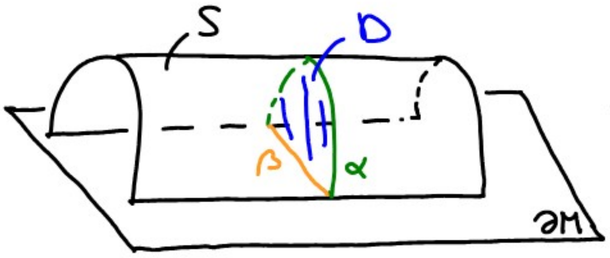
[<u>Pf:</u> See e.g. • (4) on p. 14 in §1.2 in Hatcher or
• Prop. 9.4.9 in Martelli.]

<u>Def:</u> M^3, $S \subset M$ as above. $S \subset M$ is <u>boundary compressible</u> (<u>∂-compr.</u>) if $\exists$ an essential simple arc α in S & an ess. simple arc β in ∂M s.t. $\alpha \cup \beta$ is a cld 1-mfd that bds a disk D in M w/ $\mathring{D} \subset M \setminus S$, <u>but $\nexists$ disk $D' \subset S$ w/ $\partial D' = \alpha \cup \beta'$ and $\beta' \subset \partial M$</u>.

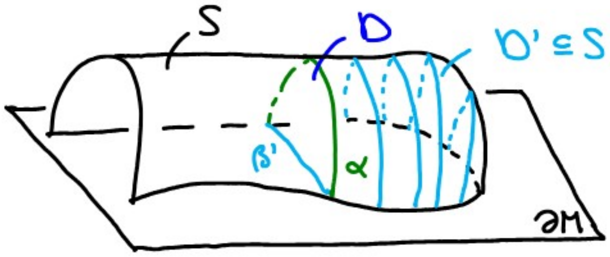
forgot to write this part on the board I think, sorry! :(
(compare w/ def'n of "compressible")

Otherwise, S is <u>boundary-incompressible</u>.

Ex:



If S is ∂-incompr., then $\exists$ $D' \subset S$ as follows:



Def: A surface $S \subset M$ is <u>boundary parallel</u> if it is separating and a component of $M \setminus S$ is $\cong_{C^0} S \times I$.

Fact: $S \subset M$ is boundary parallel $\Leftrightarrow$ S is isotopic (rel ∂) to a subsurface of ∂M. (By isotopy extension thm.)

Def: A surface $S \subset M$ is <u>essential</u> if it is incompressible, ∂-incompr. & <u>not</u> boundary parallel.

Def: An irreducible, conn., cpct, or. M^3 (w/ possibly $\partial M \neq \emptyset$) is <u>atoroidal</u> if it contains <u>no</u> essential torus.

Ex: • An irred., conn., cld, or. M^3 w/ $\boxed{|\pi_1(M)| < \infty}$ is <u>atoroidal</u> b/c $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z} \not\hookrightarrow \pi_1(M)$ in this case, so <u>spherical</u> 3-mfds, e.g. lens spaces, are atoroidal.
↳ see more below

• <u>Hyperbolic</u> 3-mfds (if you know what these are) are atoroidal (see e.g. Prop. 9.4.18, Martelli).

<u>Thm</u> (<u>Torus decomp.</u> , <u>JSJ decomp.</u>; Jaco-Shalen '76, Johannson '77)

Any closed, or., conn., <u>irred.</u> M^3 contains a finite collection of disjoint emb. incompresible tori $\mathcal{T}$ s.t. each component of $M | \mathcal{T}$ ($M | T_1 \cup \dots \cup T_r$ where $\mathcal{T} = \{T_1, \dots, T_r\}$) is either <u>atoroidal</u> or a <u>SFS</u>.

Moreover, a minimal such collection is unique up to isotopy.

We call the tori in $\mathcal{T}$ <u>JSJ tori</u>.

See e.g.
- Martelli, Sections 11.5.1 & 11.5.2
- Hatcher, Theorem 1.9
- Schultens, Theorem 3.8.2

<u>Original sources:</u>

W. H. Jaco and P. B. Shalen, Seifert fibered spaces in 3-manifolds. Mem. Amer. Math. Soc. 21 (1979), no. 220, viii+192 pp.

K. Johannson, Homotopy equivalences of 3-manifolds with boundaries, Lecture Notes in Mathematics, 761, Springer, Berlin, 1979

<u>Thm</u> (Perelman 2003):

M irreducible, atoroidal 3-mfd $\Rightarrow$ M is <u>spherical</u> or <u>hyperbolic</u>.

For more precise references, see e.g. Section 1.7 in:

M. Aschenbrenner, S. Friedl and H. Wilton. 3-Manifold Groups. Zürich: European Mathematical Society, 2015.

I will say a bit more on spherical & hyperbolic 3-mfds on the next page, but especially hyperbolic 3-mfds could be the subject of a whole new course, so this will be a very, very (!) brief account. If you want to read / learn more, I recommend e.g. the books by B. Martelli & J. Purcell (see below).

or elliptic

Spherical 3-manifolds are closed, orientable 3-manifolds that arise as quotient of S^3 by a finite subgp of $SO(4)$ ① which acts freely, smoothly and properly on S^3 (cf. Sheet 1).

Ex: • Lens spaces $L_{p,q}$

• Poincaré homology sphere P (see Lecture 10)

• For more examples, see e.g. Ex. 1.43 in Hatcher's "Algebraic Topology".

Rem: 1) Spherical 3-manifolds have finite fundamental gps ② and a "spherical geometry" ③, i.e. they admit a complete metric of constant sectional curvature +1. By Thurston's Elliptization Conj. (proven by Perelman in 2003) all three characterizations given here are equivalent. [See e.g. Conj. 12.9.3, Martelli.]

2) A list of all finite gps that arise as $\pi_1(M)$ of spherical M can be found e.g. at the beginning of Section 1.7 in M. Aschenbrenner, S. Friedl and H. Wilton. 3-Manifold Groups. Zürich: European Mathematical Society, 2015.

See also • Section 12.2, Martelli, or

• Section 4.4, W. P. Thurston and S. Levy, Three-Dimensional Geometry and Topology, Volume 1, Princeton University Press, 1997. ; Section 4.4

Hyperbolic 3-manifolds:

Similarly as for spherical 3-mfds, there are several equivalent characterizations of hyperbolic 3-mfds, e.g. a cld. 3-mfd is hyperbolic if it is the quotient $\mathbb{H}^3/\Gamma$ of hyperbolic 3-space $\mathbb{H}^3$ by a subgp Γ of isometries of $\mathbb{H}^3$ that acts freely, smoothly & properly.

A brief introduction can be found e.g. in

Schultens, Sections 7.1 & 7.2.

Much more material can be found e.g. in

- Martelli
- J. S. Purcell. Hyperbolic Knot Theory. AMS, 2020.
- W. P. Thurston and S. Levy, Three-Dimensional Geometry and Topology, Volume 1, Princeton University Press, 1997. ; Section 4.4