Lecture 13

Last time:

Thm (Torus decomp., JSJ decomp.; Jaco–Shalen '76, Johannson '77)

Any closed, or., conn., irred. M^3 contains a finite collection of

every emb. S^2 bounds a ball

disjoint emb. incompressible tori $\mathcal{J}$ s.t. each component of

$\nexists$ compressing disk, i.e.
$\forall$ s.c.c. c in $T^2 \in \mathcal{J}$ we have:
if c bounds a disk in M,
then it bds a disk in T^2 as well

Lem (Lect 12): T^2 incompr. $\Leftrightarrow$ $\pi_1(T^2) \overset{i_*}{\hookrightarrow} \pi_1(M)$ inj.

$M | \mathcal{J}$ ($M | T_1 \cup \dots \cup T_r$ where $\mathcal{J} = \{T_1, \dots, T_r\}$) is either atoroidal

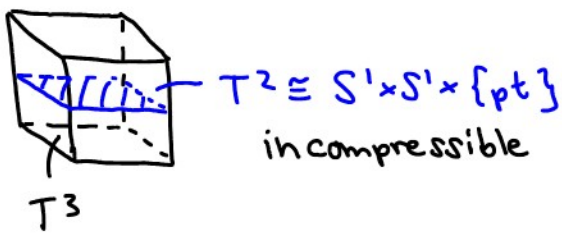
or a SFS. — see Lect 12

no essential torus

$\rightsquigarrow$ no incompressible, ∂-incompr. torus which is not boundary parallel either

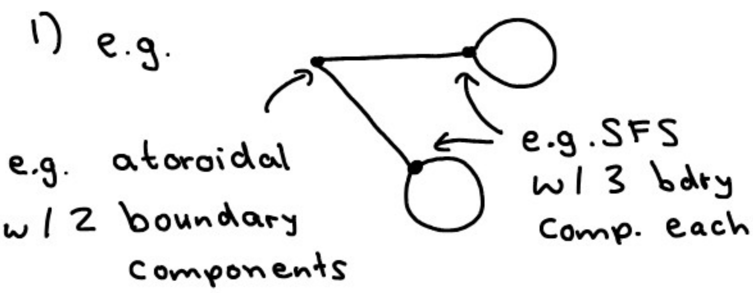
We call the tori in $\mathcal{J}$ JSJ tori.

Ex: T^3 has many incompressible tori, e.g. $S^1 \times S^1 \times \{pt\}$ $\forall pt \in S^1$



Two schematic view points on JSJ decompositions:

1) e.g.

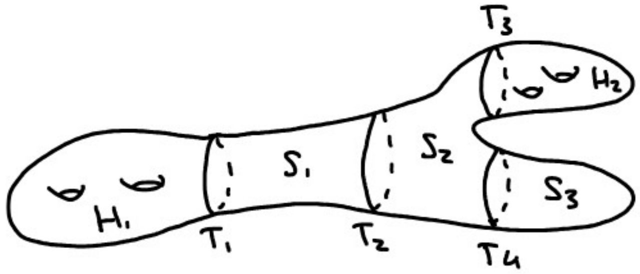


graph w/

vertices = atoroidal / SFS pieces of the JSJ decomp.

edges = JSJ tori

[Schultens, p. 96f.]

2) 

Schematic where
H_i are hyperbolic pieces,
S_i SFS,
T_i JSJ tori

<u>Concrete example</u> (w/out all the details)

← more precisely, $\mathrm{int}(N_i)$ is hyp., $i=1,2$

Let N_1, N_2 be <u>hyperbolic</u> 3-mfds, e.g. $N_1 := N_2 := S^3 \setminus \mathring{\nu} K$ for

K a <u>hyperbolic knot</u>, e.g. $K =$ fig. 8 $=$ .

(To show that $S^3 \setminus K$ is hyperbolic (i.e. admits a complete Riem. metric of constant sectional curvature -1) one can e.g.

show • that $S^3 \setminus K$ is obtained by gluing two ideal hyperbolic tetrahedra or • that $S^3 \setminus K \cong \mathbb{H}^3/\Gamma$ for

$\Gamma = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \rangle$ for $\omega^2 + \omega + 1 = 0$,
$\Gamma < PSL_2(\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$ (Riley '74))

We have $\partial N_i \cong \partial \nu K \cong T^2$, $i = 1,2$, and N_i is <u>irreducible</u> as knot exterior (see Lect 11 notes, ex. on p. 7), $i=1,2$.

Now, $\partial N_i \subseteq N_i$ is <u>incompressible</u> for $i = 1,2$ (see Ex. on p. 8 of Lect 12 notes).

Let $N := N_1 \cup_{\partial N_1 \cong \partial N_2} N_2$. Then N is irreducible and contains an incompressible torus $\partial N_1 \cong \partial N_2$. (Follows using the arguments in (4) on p. 14 in §1.2 in Hatcher / Prop. 9.4.9 in Martelli.)

One can show that the JSJ decomp. of N needs at least one JSJ torus (b/c N is neither hyperbolic nor a SFS) and indeed the torus $\partial N_1 \cong \partial N_2 \subset N$ is enough.

OUTLOOK : "what else can we do w/ 3-manifolds?"

(this outlook is <u>not</u> exam-relevant)

A few more directions to (topologically) study 3-manifolds:

- <u>open book decompositions</u>: Every cld, or., conn. 3-mfd M has an open book decomp., i.e. $\exists$ link $L \subset M$ and a fibration $M \setminus L \xrightarrow{p} S^1$ s.t. $\forall \theta \in S^1$ $p^{-1}(\theta) \cong \mathring{\Sigma}$, $\Sigma \subset M$ cpct surf. w/ $\partial\Sigma = L$.

  These open book decomp. have relations to <u>contact geometry</u>.

  <u>Key words:</u> contact structures, Giroux correspondence.

  The above existence result can be proved e.g. using the existence of a Dehn surgery presentation of M (see e.g. Rolfsen, Thm 10.K.1).

- <u>branched covers</u>: Every cld, or., conn. 3-mfd M arises as branched cover of S^3 w/ branch set a knot. (Alexander 1920)

  (Away from the branch set & its preimage, a branched cover is a covering map. For the precise def., see e.g. 10.B.1 in Rolfsen (& Thm 10.G.1) or Def. 2.4.11 in Schultens (& Thm 4.10.3).)

- <u>foliations</u> which are a partion of M^3 into e.g. conn., immersed surfaces (see e.g. Rolfsen 10.A)

- . . . there's even more! ☺

<u>Some open problems on 3-manifolds</u>:

This is a very random & personal selection, e.g. I choose the first three problems b/c they seemed (most?) relevant to the topics we studied in this class.

...to be continued...