# Lecture II

**Last week:** Intro – examples
– 3 of our main results

**Today:** Manifolds: • smth $\Leftrightarrow$ top $\Leftrightarrow$ PL
• other dimensions
• Handle decompositions

## §.2 Manifolds and handle decompositions

### §.2.1 Smooth manifolds

**Def:** A topological manifold is a second countable, Hausdorff topological space $M$ which is locally Euclidean, i.e. $\forall p \in M$ $\exists$ $p \in U \subset M$ open & $\exists$ homeo $\varphi : U \xrightarrow{\cong_{C^0}} V$


$\forall p \neq q$ $\exists U, V \subset M$ open, disj. s.t. $p \in U$, $q \in V$

where • $V \subset \mathbb{R}^n$ open or
• $V \subset \mathbb{R}^n_+$ open. $\rightsquigarrow$ $(U, \varphi)$ chart at $p$

$\mathbb{R}^n_+ = \{x_n \geq 0\}$

Ex: $\mathbb{R}^n$, $\{B_r(x) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}_+\}$ $= \{B_i\}$ countable, $B_i \subset M$ open s.t. $\forall U \subset M$ open $U = \bigcup_{i \in I} B_i$.

$n :=$ dimension of $M$. [welldef'd : inv. of domain]

A smooth structure on $M$ is a maximal (w.r.t. inclusion) smooth atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ for $M$, i.e. $\bigcup_{i \in I} U_i = M$ & any two charts $(U, \varphi)$, $(V, \psi) \in \mathcal{A}$ are smoothly compatible:

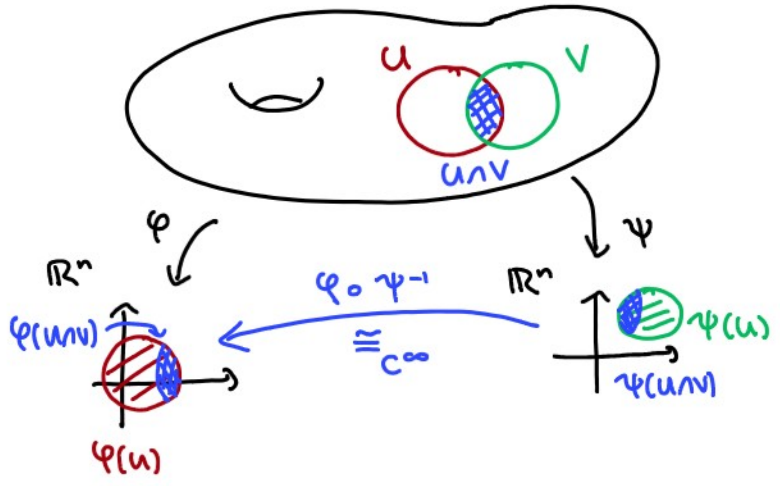
collection of charts

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \xrightarrow{\cong_{C^\infty}} \varphi(U \cap V) \text{ is a diffeomorphism.}$$

$\psi(U \cap V) \subset \mathbb{R}^n \dashrightarrow \varphi(U \cap V) \subset \mathbb{R}^n$

transition map

↳ smooth, bij. map w/ smooth inverse (as map $\mathbb{R}^n \to \mathbb{R}^n$, so all component fcts have cont. partial derivatives of all orders

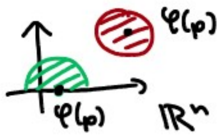



drop from notation

$\rightsquigarrow$ $M$ (together w/ $\mathcal{A}$) is a smooth manifold.

$\rightsquigarrow$ • <u>boundary</u> $\partial M = \{p \in M \mid \exists (U,\varphi)$ $\boxed{\text{boundary chart}}$ at $p$, where $\varphi(U) = V \subset \mathbb{R}^n_+$ open & $\varphi(x) \in \partial \mathbb{R}^n_+\}$

• <u>interior</u> $\mathrm{Int}(M) = \{p \in M \mid \exists (U,\varphi)$ $\boxed{\text{interior chart}}$ at $p$, $U \subset \mathbb{R}^n$ open$\}$




If $\partial M \neq \emptyset$, we call $M$ a <u>manifold w/ boundary</u>.

A compact mfd $M$ w/ $\partial M = \emptyset$ is <u>closed</u>.

<u>Ex:</u> • A <u>surface</u> is a mfd of dim. 2.

• We've seen many smooth 3-mfds last week.

Reference for smooth mfds: [Lee, Introduction to smooth manifolds].

A map $f: M \to N$ between smooth mfds $M, N$ is <u>smooth</u> if $\forall$ charts $(U,\varphi)$ of $M$, $(V,\psi)$ of $N$, the map $\psi \circ f \circ \varphi^{-1} \in C^\infty$ whereever it is defined.
<u>Exer:</u> draw corresponding schematic.

A <u>diffeomorphism</u> $M \to N$ is a smooth map w/ a smooth inverse.

⚠ For us, "manifold" will mean <u>smooth</u> manifold.
(from now on)
["mfd" = smooth mfd; top. mfds, ...]

## § 2.2 Perspectives on manifolds

In the definition of a smooth manifold, we could ask the transition maps $\varphi \circ \psi^{-1}$ to be

- $C^r$ $(C^0 = \text{top. mfd})$
- biholomorphic (if $2 \mid n$)
- ⋮
- isometries on e.g. $\mathbb{R}^n$, $\mathbb{H}^n$, $S^n$

$\rightsquigarrow$ additional geometric structure

↳ important e.g. for $n=3$:
Thurston's Geometrization Conj.
(Perelmann 2003)

- PL = "piecewise linear"
(between open sets of $\mathbb{R}^n$)

[$\rightsquigarrow$ restriction of a simplicial map def'd on polyhedron of some simplicial cx]

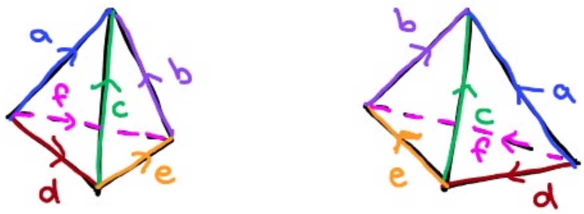
PL manifolds admit a triangulation, [PL $\Rightarrow$ triangulable]

i.e. can be represented as geometric realization of a 3-dimensional simplicial complex. [$\exists$ simplicial cx $X$ = finite collection of simplices s.t. faces of simplices are simpl. & intersections of simpl. are faces of both simpl. or $\emptyset$; union of all simpl. $\rightsquigarrow |X|$; $M$ triangulated by $X$ if $\exists$ homeo $|X| \xrightarrow{\cong} M$]

n=3:  3-simplex = tetrahedron $\rightarrow$ building block of 3D simpl. cx.

**Thm** (Bing '59, Moise '52 '50s):
Every compact topological 3-mfd admits a triangulation & has a PL structure.

**Ex:** $S^3$:




**Thm** (Munkres '60): Every compact topological 3-mfd has a unique smooth structure (up to diffeomorphism).
also Whitehead '61
$(M,\mathcal{A}) \to (M,\mathcal{B})$

See also [Thurston, Three-dimensional Geometry and Topology, Section 3.10].

**Rem:** DIFF $\subseteq$ PL $\subseteq$ TOP
(category of smooth mfds) Whitehead '40

[TRIANG $\not\Rightarrow$ PL
Ex: double suspension of Poincaré homology sphere]
source

$n = 1,2,3$: DIFF = PL = TOP
But in general $\neq$ $\neq$

$n = 4$: DIFF = PL $\neq$ TOP

($n \leq 6$: SMOOTH $\Leftarrow$ PL) [Hirsch–Mazur '74]

[See also [Munkres Elementary Differential Topology, Chapter II]]

## Examples of weird phenomena:

**n=7** [Milnor '56]: $S^7$ has exotic smooth structures, [Kervaire-Milnor $\Rightarrow$ 28 smooth structures]
i.e. $\exists$ smooth $M^7$ s.t. $M \cong_{C^0} S^7$, $M \not\cong_{C^\infty} S^7$.

**n=4:** $\exists$ uncountably infinitely many smooth structures on $\mathbb{R}^4$.
[Gompf, Taubes '80s]
'83, '85 '87

In contrast, if $M^n \cong_{C^0} \mathbb{R}^n$, then $M \cong_{C^\infty} \mathbb{R}^n$.

### Poincaré conjecture

(goes back to Conj. by Poincaré 1904)
Henri Poincaré, French "universalist" 1854–1912

Let $M^n$ be a closed mfd, top. or smooth (TOP/DIFF),

homotopy equivalent

s.t. $\boxed{M \simeq S^n}$. Then $M \cong_{C^0/C^\infty} S^n$.

[Short version in TOP: Homotopy spheres are homeomorphic to spheres.]

**TOP**

$\boxed{\text{YES!}}$

n=1 ✓
n=2 ✓ classif. of surf.
n=3 Perelman 2003
[Ricci flow, Hamilton, ...
long history: Poincaré homology sphere, Whitehead mfd, ...]
$H_*(P) = H_*(S^3)$, $|\pi_1(P)| = 120$
open, contractible $M^3$, noncpct, $\partial M = \emptyset$, $M^3 \not\cong_{C^0} \mathbb{R}^3$

**Diff**

↳ maybe :)

n=1,2,3 $\boxed{\text{YES!}}$
n=7 $\boxed{\text{NO!}}$ Milnor 1956
$n \geq 5$ mostly well-understood
[true for n=5, 6, 12, 56, 61]

$\boxed{n=4}$ Open!

n=4 Freedman 1982; book "Disk Emb. Thm" (editors: Behrens-Kalmar-Kim-Powell-Ray '21)
$n \geq 5$ Smale 1960
↳ handle decompositions!

## § 2.3 Handle decompositions

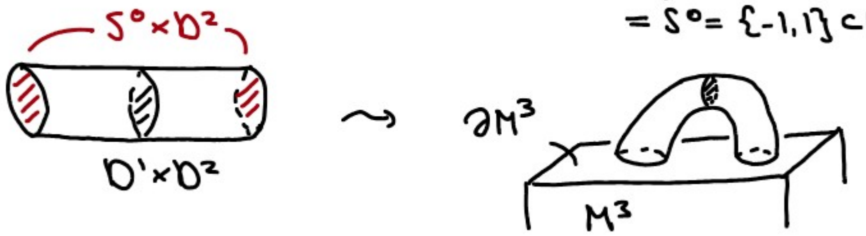
Let $M^n$ be a (smooth!) manifold.

**Def:** For $0 \leq k \leq n$, an $n$-dimensional $k$-handle $h$ is a copy of $D^k \times D^{n-k}$, attached to $M$ via an embedding (diffeo onto its image) $\varphi: \partial D^k \times D^{n-k} \hookrightarrow \partial M$.

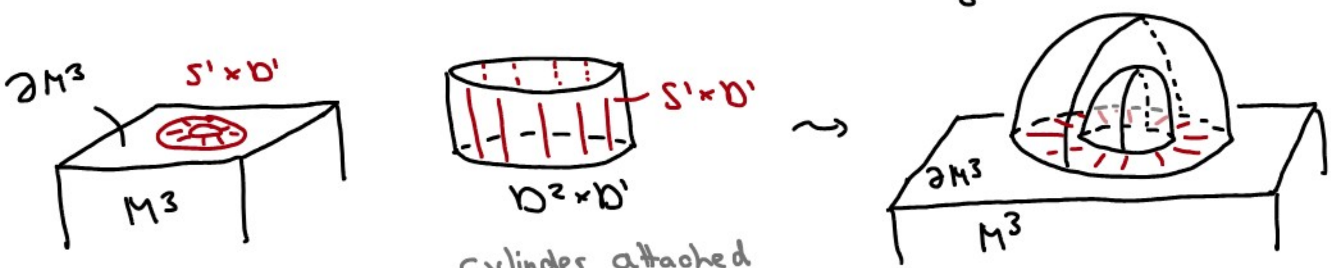
**Ex:** $k=0$: A $0$-handle is $\underset{\{0\}}{\overset{\shortparallel}{D^0}} \times D^{n-k}$ attached along $\varphi: \partial D^0 \times D^n = \emptyset$, so just disjoint union w/ $D^n$.

$\boxed{n=3}$ $k=0$:

$k=1$: 1-handle $D^1 \times D^2$ att. along $\varphi: \underbrace{\partial D^1}_{= S^0 = \{-1,1\} \subset \mathbb{R}} \times D^2 \hookrightarrow \partial M$




$k=2$: 2-h. $D^2 \times D^1$ att. along $\varphi: \underbrace{\partial D^2}_{= S^1} \times D^1 \hookrightarrow \partial M$.




cylinder attached along annular part of its boundary

$k=3$: 3-h is $D^3 \times \{0\}$ att. along $\varphi: \underbrace{\partial D^3}_{= S^2} \times \{0\} \hookrightarrow \partial M$

fill in $S^2$ in $\partial M$
[not possible if e.g. $M = S^1 \times D^2$, $\partial M = S^1 \times S^1$]

**Rem:** Note that $D^k \times D^{n-k} \cong_{C^0} D^n$

and $\partial(D^k \times D^{n-k}) = \partial D^k \times D^{n-k} \cup D^k \times \partial D^{n-k}$.

$\rightsquigarrow$ we're really gluing $n$-balls along parts of their boundary

**Def:** For an n-dim. k-handle $h_k$ attached to $M$ via $\varphi: \partial D^k \times D^{n-k} \hookrightarrow \partial M$,

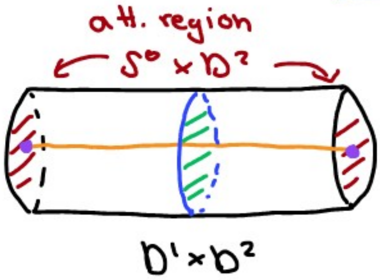
$\partial D^k \times D^{n-k}$ (or $\varphi(\partial D^k \times D^{n-k})$) is called attaching region

$D^k \times \{0\}$ is called core

$\partial D^k \times \{0\}$ -"- attaching sphere

$\{0\} \times D^{n-k}$ -"- cocore

$\{0\} \times \partial D^{n-k}$ -"- belt sphere




Cocore $\{0\} \times D^2$ w/ bdry the belt sphere $\{0\} \times S^1$

Core $D^1 \times \{0\}$ w/ att. sph. $S^0 \times \{0\}$ as bdry

**Rem:** View/ think of $M \cup_\varphi h$ as smooth manifold.

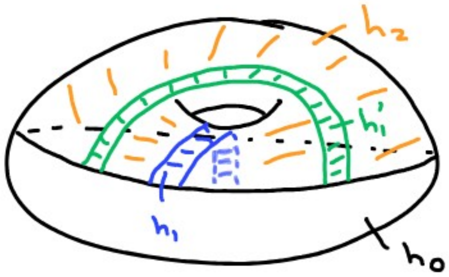
⇝ actually mfd with corners, but can "straighten" smoothen them (in any dimension!) [Wall, Differential Topology, §2.6]

in dim $n=3$: $\exists!$ smooth structure (see above)

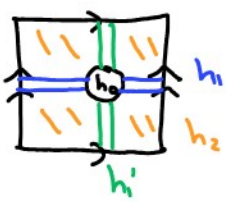
**Ex:**

- $S^n =$  $= D^n \cup D^n = h_0 \cup h_n$

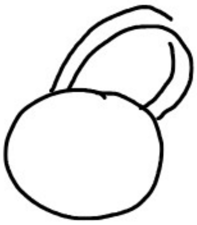
- $T^2 = S^1 \times S^1$




$=$




$=$


- $\mathbb{RP}^2 =$  $\cup\, h_2$

- $S^2 =$


⇝ handle decompositions not unique

**Def:** $\varphi_0, \varphi_1 : Y \hookrightarrow X$ embeddings are <u>isotopic</u> if $\exists$ smooth homotopy $h: Y \times I \to X$ <u>through</u> embeddings, i.e. $h_t : Y \to X$ is an emb. $\forall\ 0 \le t \le 1$.
$h_t(\cdot) = h(\cdot, t)$, $h_0 = \varphi_0, h_1 = \varphi_1$

**Thm** (<u>Isotopy Extension Theorem</u>) [Wall, Thm 2.4.2] [Kosinsky, Thm 5.2 in Chapter II]
(Thom, Cerf, Palais) [Hirsch, §8.1]

$Y, X$ compact & $h: Y \times I \to X$ isotopy of embeddings $Y \hookrightarrow X$
$\Rightarrow\ \exists\, H: X \times I \to X$ smooth map s.t.

- $H_0 = id_X$
- $H_t : X \to X$ $\quad (H_t(\cdot) = H(\cdot, t))$ diffeo $\forall\, 0 \le t \le 1$
- $h_t = H_t \circ h_0 \quad \forall t$

} ambient isotopy [or diffeotopy]

(locally flat) [top. version due to Edwards–Kirby]

[Pf idea: vector fields]

**Lemma:** $\varphi_i : \partial D^k \times D^{n-k} \hookrightarrow \partial M$, $i = 1,2$ isotopic embeddings
$$\Rightarrow M \cup_{\varphi_1} h_k \cong_{C^\infty} M \cup_{\varphi_2} h_k.$$
Will repeat in Lect. 3

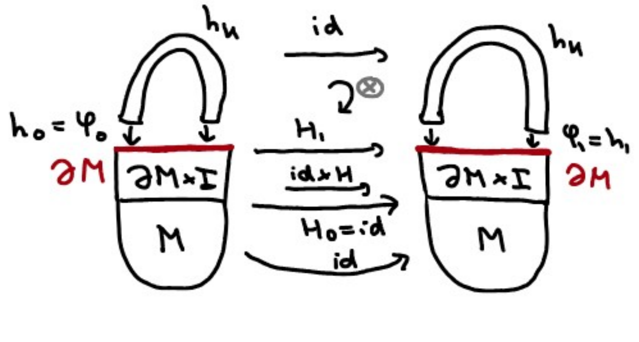
**Pf:** Extend an isotopy $h = \varphi_t$ to an ambient isotopy $H: \partial M \times I \to \partial M$ using the Isotopy Extension Theorem.

Use this to find a diffeo $M_0 \dashrightarrow M_1$:

$$\begin{array}{ccccc} M_0 & = & M & \cup_{\varphi_0} & h_k \\ \downarrow & & \downarrow H_1 & & \downarrow id \\ M_1 & = & M & \cup_{\varphi_1} & h_k \end{array}$$

To make this more precise, consider the diffeo
$$H \times Id_I : \partial M \times I \to \partial M \times I$$
Now, identify $\partial M \times I$ w/ a collar neighborhood of $\partial M$ in $M$, i.e. a nbhd of $\partial M$ in $M$ diffeom. to $\partial M \times I$ s.t. $\partial M$ is identified w/ $\partial M \times \{0\}$. ⊗ [or, put differently, an emb. $\partial M \times I \hookrightarrow M$ ext. $\partial M \times \{0\} \xrightarrow{proj} \partial M$]

**Schematic:**




⊗ $H_1 \circ h_0 = h_1 \circ id$

□

⊗ **Lemma:** For every mfd w/ bdry, the bdry has a collar nbhd. [Wall, Differential Topology, Thm 1.5.5]
[Pf idea: vector fields]