Lecture IV

Last week:
- Handles can be rearranged s.t. $M = h^0 \cup h_1^1 \cup \ldots \cup h_1^{k_1} \cup h_2^1 \cup \ldots$ (increasing indices) and we can assume that there is a unique 0- & n-handle
- handle cancellation

§ 3. Heegaard splittings

§ 3.1 Existence

Def: Let M^n be a (smooth) manifold. An orientation on M is a (maximal) smooth atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ on M s.t. $\forall p \in M$ $\forall i,j$:

$$\det(\underbrace{J_q(\varphi_i \circ \varphi_j^{-1})}_{\substack{\text{Jacobian matrix:} \\ \frac{\partial f_i}{\partial x_j} \text{ in local coordinates}}}) > 0 \quad \forall q \in \varphi_j(U_i \cap U_j).$$

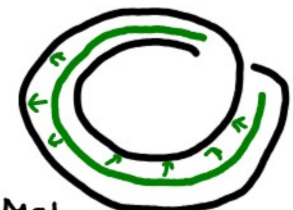
M is orientable $:\Leftrightarrow$ it has an orientation. Otherwise M is nonorientable.

$(M, \mathcal{A})$ w/ $\mathcal{A}$ orient. is oriented.

Rem: 1) There are many (!) equivalent definitions of orientations & orientability, see e.g. "Orientations of Manifolds" on "The Manifold Atlas Project". http://www.map.mpim-bonn.mpg.de/Orientation_of_manifolds

2) "Working def'n": (w/out pf)
- A surface is orientable $\Leftrightarrow$ it does not contain a Möbius band.

$$\text{Möb} = \frac{[0,1] \times [0,1]}{(s,0) \sim (1-s,1)}$$



- A 3-mfd is orientable $\Leftrightarrow$ it does not contain $\text{Möb} \times [0,1]$. [Rolfsen, 2G]

Also equivalent: "consistent" choice of "Right-Hand-Rule" or "↺"

[**Rem:** $\partial(\text{Möb} \times I) = \text{Klein bottle} = \text{Möb} \cup_\varphi \text{Möb}$]

Def: A smooth map $f: M \to N$ for oriented $(M, \{(U_i, \varphi_i)\})$, $(N, \{(V_j, \psi_j)\})$

is orientation-preserving if $\det(J_q(\psi_j \circ f \circ \varphi_i^{-1})) > 0$

orientation-reversing -"- -"- < 0

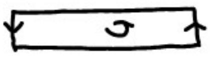
whenever its defined.

Ex: If M, N are oriented manifolds & $\varphi: X_1 \subset \partial M \to X_2 \subset \partial M$ is

an or.-reversing diffeo, then $M \cup_\varphi N$ is oriented.

Ex:

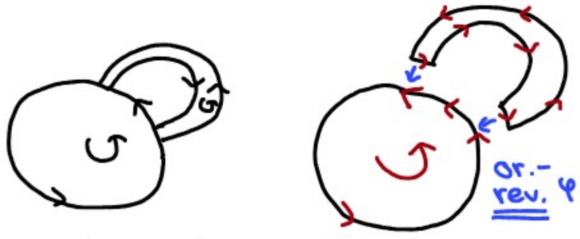
oriented 2-dim. 0-h. D^2



2-dim 1-h. $D^1 \times D^1$

2 possibilities

(I)



1-h. attached along

or.-rev. emb.

$S^0 \times D^1 \hookrightarrow \partial(0\text{-h})$

$\rightsquigarrow$ annulus orientable

(II)

Möb (Möbiusband)

nonorientable

~~here: 1-h. att along or.-pres. emb. $S^0 \times D^1 \hookrightarrow \partial D^2$~~ no!

here: att. map neither or. pres. nor or. rev.

Correction: When we glue a 2-dim 1-handle to a 0-handle, there are 2 possibilities as described on the right: we can obtain the orientable annulus or the nonorientable Möbiusband. However, this distinction does not come from the distinction whether the attaching map is or. rev. or or. preserving. If we glue the oriented strip (1-h. $D^1 \times D^1$) to the or. disk D^2 via an or. preserving att. map, we also get an annulus, but non-oriented.

Rem: Above def'n of orientation makes sense for mfds w/ boundary. However, other def'ns might sometimes be more convenient.

An orientation on a mfd M w/ ∂ induces an orientation on ∂M, e.g. by restricting charts.

(See e.g. Section 8 of http://www.map.mpim-bonn.mpg.de/ .)

Another possibility is to work w/ orientations of tangent spaces (as vector spaces) & use the convention "outward normal first" (see e.g. Section §3.2 of [Guillemin-Pollack] or Chapter 15 (in part. Prop. 15.24) in [Lee].)

Def: A <u>handlebody</u> H is a compact, connected, orientable 3-mfd w/ boundary whose handle decomposition consists only of 0- & 1-handles.

<u>genus</u> $\quad \text{genus}(H) := g(\partial H)$

closed surface $\quad \Sigma_g =$ 

Prop 1: Any handlebody of genus g is diffeomorphic to $\natural_g (S^1 \times D^2)$.

Cor: Any two handlebodies of the same genus are diffeo to each other.

Rem: There is an analog definition for n-mfds $(n \geq 2)$. $\rightsquigarrow \natural_g (S^1 \times D^{n-1})$

[$\rightsquigarrow$ n-dim 1-handlebody]

[to stress that there are just 0- & 1-handles]

Def (<u>boundary connected sum</u>): For M_1^n, M_2^n or. mfds w/ boundary, (oriented)

$$M_1 \natural M_2 := M_1 \cup_\varphi M_2 \quad \text{where } \varphi : D_2^{n-1} \subset \partial M_2 \xrightarrow[\text{or. rev.}]{\cong_{C^\infty}} D_1^{n-1} \subset \partial M_1.$$

(+ smoothen corners).

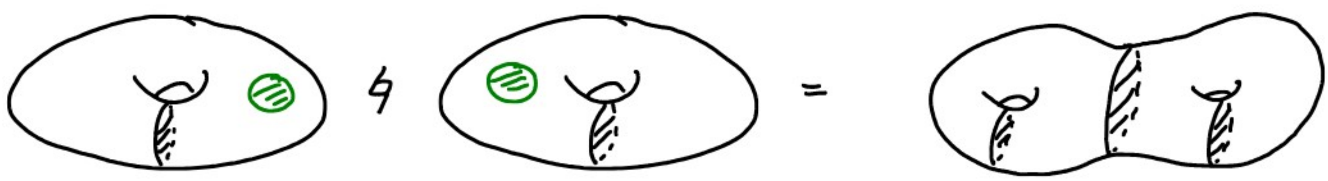
[$\natural$ is called "<u>boundary sum</u>" e.g. in [Wall] or [Gompf-Stipsicz]; sometimes also denoted $\#_\partial$.]

Rem: This is <u>well defined</u> by the <u>Disk Theorem</u> (Palais 1960) & independent of all choices. [see e.g. Wall, Prop. 2.7.5] (we choose φ or. <u>rev.</u>!)

<u>Thm</u> (Disk Theorem): Any two emb. $D^n \hookrightarrow M^n$ either both or. pres. or both or. rev. are ambiently isotopic.

[see e.g. Wall, "Diff. Topology", Thm 2.5.6]

Ex: <u>$n = 3$:</u>



Note that $\partial(M^3 \natural N^3) = (\partial M)^2 \# (\partial N)^2$, ← connected sum

e.g. $\partial(\natural_g (S^1 \times D^2)) = \#(S^1 \times S^1) = \Sigma_g$.

[true also for other n, but there we haven't defined $\#$ yet.]

Sketch of proof of Prop 1: here more details than in lecture!

(for general $n \geq 2$; we only need $n=3$!)

[see also Gompf-Stipsicz, p. 101, Ex. 4.1.4 (b))]

Let H be a handlebody, in particular H is conn., cpct, orientable.

By Prop. from Lecture 3 & the def'n of handlebodies ($\rightsquigarrow$ only 0- & 1-handles),

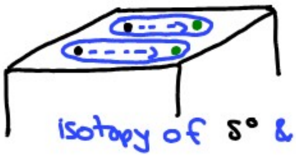
H has a handle decomposition $H = h_0 \cup h_1^1 \cup \dots \cup h_1^g$.

Consider the attaching map $\varphi_i : \partial D^1 \times D^{n-1} \hookrightarrow \partial D^n$ (0-handle h_0) of the i-th 1-handle h_1^i.

(reordering lemma allows us to think of handles being attached simultaneously!)

Note that any two embeddings $\varphi_{i,0}, \varphi_{i,0}' : \underbrace{\partial D^1}_{= S^0 \text{ (two pts)}} \times \{0\} \hookrightarrow \partial D^n$

are isotopic.



The embedding φ_i is determined by the embedding $\varphi_{i,0} : \partial D^1 \times \{0\} \hookrightarrow \partial D^n$

together w/ a framing of $\varphi_{i,0}(\partial D^1) \subset \partial D^n$. (see also rem. in Lect. 3)

$\cong \partial D^1 \times \{0\}$ [slight abuse of notation]

Framings of $K := \varphi_{i,0}(\partial D^1) \subset \partial D^n$ are homotopy classes of maps

$K \cong S^0 \to GL_{n-1}(\mathbb{R})$, i.e.

$n=3$: $GL_2(\mathbb{R})$

$\{\text{framings}\}/_{\text{homotopy}} \cong \pi_0(GL_{n-1}(\mathbb{R})) \cong \text{conn. comp. of } GL_{n-1}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$

$\rightsquigarrow$ two possible framings up to homotopy

Since H is orientable, there is a unique framing of K s.t.

attaching h_1^i makes $D^n \cup h_1^i$ orientable.

[e.g. for $n=2$: $h_0 \cup h_1 \rightsquigarrow$ annulus, Möb]

[Check that $\natural_g (S^1 \times D^{n-1})$ is orientable and has a handle decomp. consisting of one 0-h & g 1-h.]

$\square$

Thm 1 (Heegaard splitting)

Every closed, connected, orientable 3-manifold M has a Heegaard splitting, i.e. $\exists g \geq 0$ & there exist handlebodies H_1, H_2 of genus g, $\exists \varphi: \partial H_2 \xrightarrow[\text{or. rev.}]{\cong_{C^\infty}} \partial H_1$ s.t. $M = H_1 \cup_\varphi H_2 = \dfrac{H_1 \sqcup H_2}{x \sim \varphi(x)}$.

We will sometimes just write $M = H_1 \cup_{\partial H_1} H_2$ (sloppy!).

<u>Proof:</u> Let M^3 be as in the thm. By what we learnt in §2.3 on handle decompositions, M admits a handle decomp. w/ a unique 0-h & a unique 3-h in which handles are attached w/ increasing indices (0-h before 1-h, then 2-h, then 3-h).

$$\rightsquigarrow \quad M = \underbrace{h_0 \cup h_1^1 \cup \ldots h_1^g}_{=:H_1} \cup h_2^1 \cup \ldots \cup h_2^e \cup h_3$$

By def'n, H_1 is a handlebody (of genus g).

Define $H_2 := M \setminus \mathrm{Int}(H_1)$.

By dual handle decomp.,

$H_2 \cong \tilde{h}_0 \cup \tilde{h}_1 \cup \ldots \cup \tilde{h}_1^e$, so H_2 is also a handlebody.

By Prop. 1 above, $H_1 \cong_{C^\infty} \natural_g (S^1 \times D^2)$,

$H_2 \cong_{C^\infty} \natural_e (S^1 \times D^2)$.

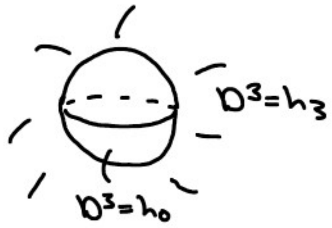
Since $\begin{array}{ccc} \partial H_1 & = & \partial H_2 \\ \wr\wr & & \wr\wr \\ \Sigma_g & & \Sigma_e \end{array}$ [M closed $\Rightarrow \partial H_2 = \partial(M \setminus \mathrm{Int}\, H_1) = \partial H_1$]

($\Rightarrow \Sigma_g \cong \Sigma_e$; both closed surf.)

we obtain $g = e$ by the classification of surfaces. $\square$

[<u>Rem:</u>
- If M^3 has a Heegaard splitting, then M is closed & orientable.
- also works for mfds w/ boundary $\rightsquigarrow$ "compression bodies" instead of handlebodies. [see Schultens, 6.7.2]]

Ex: • $S^3 = D^3 \cup_{S^2} D^3$
using $-id_{S^2}$



• $S^3 = S^1 \times D^2 \cup_\varphi D^2 \times S^1$ (Exer Sheet 1, Exer 1)

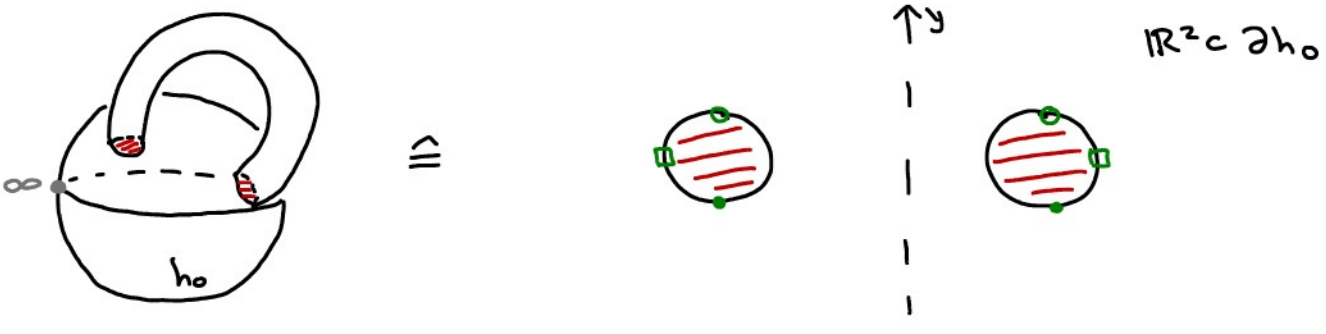
§3.2 Heegaard diagrams

Q: How can we represent Heegaard splittings?
A: Heegaard diagrams.

Let $M = \underbrace{h_0 \cup h_1^1 \cup \ldots \cup h_1^g}_{H_1} \underbrace{\cup h_2^1 \cup \ldots \cup h_2^g \cup h_3}_{H_2}$ be a Heegaard splitting.

Consider $\partial h_0 = \partial D^3 = S^2 \cong \mathbb{R}^2 \cup \{\infty\}$. [$\leadsto \mathbb{R}^2$ "$\subset$" ∂h_0 by this diffeo [really $\mathbb{R}^2 \hookrightarrow \partial h_0$ emb.]]

We can view the attaching region of a 1-handle h_1^i as
$S^0 \times D^2 = D^2 \sqcup D^2 \subset \mathbb{R}^2 \subset \partial h_0$. (assuming it misses the pt at ∞)



identify D^2's via $(x,y) \mapsto (-x,y)$ (reflection).

Attaching a 1-handle to h_0 is equivalent to gluing two disks $D^2 \sqcup D^2 \subset \mathbb{R}^2 \subset \partial h_0$ via an orientation-reversing diffeomorphism.

WLOG we choose the reflection $(x,y) \mapsto (-x,y)$.

Next step: how do we represent 2-handles?

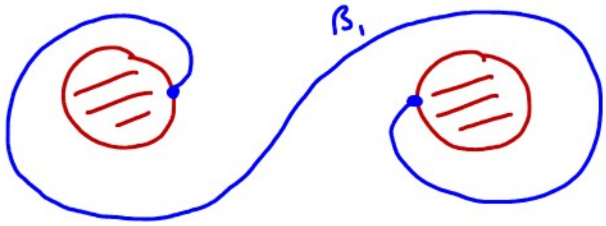
The attaching sphere of a <u>2-handle</u> is $\partial D^2 \times \{0\} \cong S^1 \subset \partial(h_0 \cup \{h_i^1\})$,

which appears in our picture / Heegaard diagram as

i.e. circles & arcs $\beta_i \subset \mathbb{R}^2$ w/ endpoints on ∂D^2, the

attaching regions of the 1-handles. (the arcs become circles S^1 after identification of the disks via reflection)

<u>Ex:</u>



$\subset \mathbb{R}^2 \subset \partial h_0$

$\cong$

What about the 3-handle? $\rightsquigarrow$ later!

<u>Def:</u> $\mathbb{R}^2$ together w/ att. regions $(D^2 \sqcup D^2)$'s of the 1-handles & the att. spheres β_i (arcs or circles) of the 2-handles is called a (<u>planar</u>) <u>Heegaard diagram</u>.

⚠ Note that w/ this def'n not every Heegaard diagram would give a 3-mfd. We make more precise which additional conditions we need on the pairs of disks & β_i in Lecture 5.

(e.g. there should be g attaching circles & g pairs of disks $D^2 \sqcup D^2$).

<u>Rem:</u> Sometimes $(\partial H_1, \beta_i)$ is called a <u>Heegaard diagram</u>.

[Note that in the literature, there is also yet another version where any surface Σ_g is drawn together w/ g att. circles $\{\beta_i\}$ for H_2 & g att. circles $\{\alpha_i\}$ for H_1.]

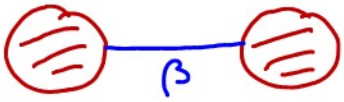
<u>References on Heegaard diagrams (& Heegaard splittings):</u>

- Gompf - Stipsicz, Section 4.3
- Rolfsen, Section 9.C
- Schultens, Section 6.2
- Prasolov - Sossinsky, § 10
- Matsumoto, Section 5.2(b)

Ex: 1) $S^3 = h_0 \cup_{-id} h_3$: $\emptyset \subset \mathbb{R}^2$ ← Heegaard diagram for S^3

2) $S^3 = S^1 \times D^2 \cup D^2 \times S^1 \cong h_0 \cup h_1 \cup h_2 \cup h_3$ (*)

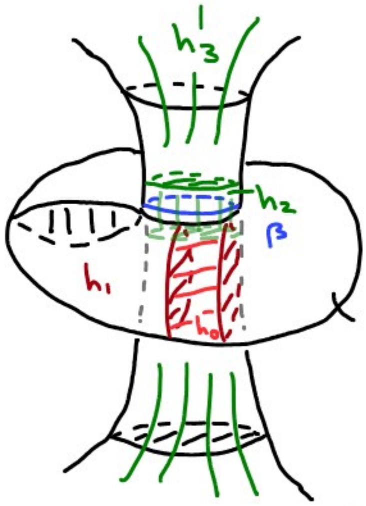
Corresponding Heegaard diagram:



Equivalently,

∂H_1 $(=\partial H_2)$ $= S^1 \times S^1$ (here)

is also a Heegaard diagram describing the Heegaard splitting (*)



$h_2 = D^2 \times D^1$

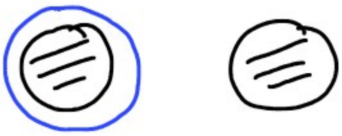
$h_3 = D^3$

$h_0 = D^3$, $h_1 = D^1 \times D^2$

$H_1 = S^1 \times D^2 = h_0 \cup h_1$

$H_2 = D^2 \times S^1 = h_2 \cup h_3$ $(\cong \tilde{h}_0 \cup \tilde{h}_1)$

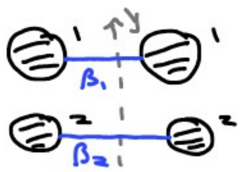
3) Heegaard diagram for $S^1 \times S^2 = S^1 \times D^2 \cup S^1 \times D^2$:



Equivalently,

∂H_1

4) Another Heegaard diagram for S^3:

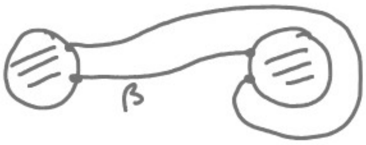


(recall: we identify pairs of disks by reflection along y-axis)

5) **Exer:** Find a Heegaard diagram for $T^3 = S^1 \times S^1 \times S^1$.

(Exercise Sheet, Exer 3)

6)



$\rightarrow \mathbb{R}P^3 = L_{2,1}$

← one curve β

(which becomes att. circle ($\cong S^1$)

after identification of the disks)

Outlook:

Thm 2: A Heegaard diagram determines a unique 3-manifold up to diffeomorphism. → + extra conditions, see Lecture 5 !

Pf: See Lecture 5 (sketch).