Lecture 6

Oral exam dates:
- Feb 10-14, 2025
- March 17-19, 2025

Please let me know ahead of time (end of January) if you have special needs for the exam or have time conflicts with other exams on certain dates / at certain times. We'll schedule the precise times at the end of the semester.

Next week:
- Aru Ray
- evaluation

Dec 4: No lecture (Dies academicus)

Last two lectures:

§. 3 Heegaard splittings

§. 3.1 Existence

§. 3.2 Heegaard diagrams

$M = H_1 \cup_{\partial H_1 \cong \partial H_2} H_2$

{Heegaard diagrams} $\longrightarrow\!\!\!\!\rightarrow$ {Heegaard splittings}

$\Downarrow$

or. conn. cld M^3

1:1

Reidemeister-Singer

preserve genus of ∂H_1: isotopies, handle slides

genus $g' = g \pm 1$: (de-) stabilizations

Today: § 3.3 Heegaard genus, lens spaces

§ 3.3 Heegaard genus

Def: The Heegaard genus of an (or., closed, conn.) M^3 is

$$g(M) := \min\{ g(\Sigma) \mid M^3 = H_1 \cup_\Sigma H_2 \text{ Heegaard splitting of } M \}.$$

$[0 \le g(M) < \infty]$ Σ: Heegaard surface of the Heegaard splitting

Prop: $g(M^3) = 0 \iff M \cong_{C^0} S^3$. $(\iff M \cong_{C^\infty} S^3)$
[Bing, Moise, Munkres; see Lect 2]

Pf: "$\Leftarrow$": ✓ $(S^3 = D^3 \cup_{S^2} D^3)$

"$\Rightarrow$": Use Exer 4c, Exercise Sheet 2. □

Cor: $g(S^1 \times S^2) = 1$.

Pf: We saw "$\le$": $S^1 \times S^2 \,\hat{=}$ 

$[\;= S^1 \times D^2 \cup_\varphi S^1 \times D^2$, note that $D^2 \cup_{\varphi|} D^2 = S^2.]$

"$\ge$" follows from the prop. using e.g. $\pi_1(S^1 \times S^2) \cong \mathbb{Z} \not\cong \{1\} = \pi_1(S^3)$.

[Suppose $g(S^1 \times S^2) = 0 \overset{\text{Prop}}{\Longrightarrow} S^1 \times S^2 \cong_{C^0} S^3$ ↯ to ↑
b/c homeomorphic mfds have isomorphic π_1]
$(X \cong_{C^0} Y \Rightarrow X \underset{\text{h.e.}}{\simeq} Y \Rightarrow \pi_1(X) \cong \pi_1(Y))$

Recall: Lens space $L_{p,q} := S^3/\sim$ where [studied by Tietze in 1908]

$S^3 = \{ (z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1 \}$, $p, q \in \mathbb{Z}$ coprime, $\boxed{p \neq 0}$,
$(\gcd(p,q) = 1)$

and $[m] \circ (z,w) := (e^{2\pi i m/p} z,\; e^{2\pi i q m/p} w)$ for $[m] \in \mathbb{Z}/p\mathbb{Z}$.

(action of $\mathbb{Z}/p\mathbb{Z}$ on S^3 generated by
$(z,w) \sim 1 \circ (z,w) = (e^{2\pi i/p} z,\; e^{2\pi i q/p} w)$.)

[See Lect. 1 & Exer Sheet 1, Exer 2.]
↳ this action is free, smooth & proper
⇝ quotient $L_{p,q}$ is (smooth) 3-mfd

Ex:

- $L_{1,0} \cong_{C^\infty} S^3$, $L_{2,1} \cong_{C^\infty} \mathbb{RP}^3$ (see Exer 2).
  $(\cong S^3/_{(z,w) \sim (-z,-w)})$
- $L_{1,q} \cong_{C^\infty} S^3 \;\forall q$ (exer).
- Indeed, $L_{p,q+kp} \cong L_{p,q}$ $\forall k \in \mathbb{Z}$ (b/c $e^{2\pi i k} = 1$.)

<u>Thm 1:</u> $g(M^3 = 1) \iff M$ is homeomorphic to a lens space $L_{p,q} \not\cong_{C^0} S^3$ or $S^1 \times S^2 =: L_{0,1}$.

A little more context before we prove this theorem:

- Mfds M^3 w/ $g(M^3) \geq 2$ are much less well-understood

<u>Ex:</u> $g(T^3) = 3$: Using Exer 3, Sheet 2 (hard?), you can show that $g(T^3) \leq 3$.

$$T^3 = S^1 \times S^1 \times S^1$$

<u>Exer:</u> Use π_1 (or H_1) to show "$\geq$".
(Exercise Sheet 3)

- <u>Why are lens spaces cool?</u>

<u>Fact / Exer:</u> From Sheet 1, we know $\pi_1(L_{p,q}) \cong \mathbb{Z}/p\mathbb{Z}$.

Indeed, the homotopy groups π_k and homology groups H_k of $L_{p,q}$ only depend on p!

$\ulcorner$ • $\pi_k(L_{p,q})$ indep. of q: $S^3 \to L_{p,q} = S^3/\mathbb{Z}/p$ is the universal covering map, LES (long exact sequence) of homotopy groups $\Rightarrow \pi_k(L_{p,q}) \cong \pi_k(S^3)\ \forall k \geq 2$.

(⇝ fibration)

• Fact / exer: π_1 determines $H_k\ \forall k$ for any closed, conn., or. M^3
(Sheet 3)

$$\Rightarrow H_k(L_{p,q}) \cong \begin{cases} \mathbb{Z}, & k = 0, 3 \\ \mathbb{Z}/p\mathbb{Z}, & k = 1 \\ \{0\}, & k = 2, k \geq 4 \end{cases}$$ $\lrcorner$

$\Rightarrow L_{p,q} \simeq L_{p',q'}$ only possible if $p = p'$.

$$\underset{\text{homotopy equivalence}}{\uparrow}$$

In fact, there are the following classifications:

<u>Thm 2</u> (Whitehead 1941, Reidemeister 1935, Brody 1960):
↳ homotopy equiv. ↳ PL homeo ↳ homeo

- $L_{p,q} \simeq L_{p',q'} \iff p = p'$ and $\pm q q' = m^2 \pmod p$ for some $m \in \mathbb{N}$

  $\simeq$: homotopy equivalence

  ($\pm qq'$ quadratic residue mod p)

  [This can be shown using the "torsion linking pairing".]

- $L_{p,q} \cong_{C^0} L_{p',q'} \iff p = p'$ and $q' = \pm q^{\pm 1} \pmod p$

  or $\cong_{C^\infty}$ (equiv.!)

  ["$\Leftarrow$" Exer.

  "$\Rightarrow$" e.g. using "Reidemeister torsion"; see e.g. book by V. Turaev, "Torsions of 3-dimensional manifolds"]

<u>Ex</u> (Cor. of this Thm) $L_{7,1} \simeq L_{7,2}$ (b/c $1 \cdot 2 = 3^2 \bmod 7$),

but $L_{7,1} \not\cong_{C^0} L_{7,2}$ (b/c $2^{-1} = 4 \bmod 7$).

$\rightsquigarrow$ "<u>geometric topology</u> of lens spaces differs from their <u>algebraic topology</u>."

<u>Other ex:</u> $L_{5,1} \not\cong_{C^0} L_{5,2}$ [$2^{-1} = 3 \bmod 5$)

Studied by Alexander in 1919 and historically first ex. of 3-mfds M_1, M_2 w/ isom. π_1 & homology groups, but $M_1 \not\cong_{C^0} M_2$

(Now we know that $L_{5,1} \not\simeq L_{5,2}$ which of course implies $L_{5,1} \not\cong_{C^0} L_{5,2}$.)

<u>Rem/ "fun fact":</u> Using number theory, one can determine how many different homotopy types there are among

$L_{p,1}, L_{p,2}, \ldots, L_{p,p-1}$ for any $p > 0$ (if p is not a prime, consider only $L_{p,q}$ w/ $\gcd(p,q) = 1$).

<u>Ex:</u> [See Rolfsen, Rem. 9.B.7.] For prime p, there

- are precisely two homotopy types if $p \equiv 1 \pmod 4$
- is exactly one —"— type if $p \equiv 3 \pmod 4$.

Pf of "$\Leftarrow$" of Thm 1:

We need to show that $g(L_{p,q}) = 1$ for $p \neq 1$.

Recall from Exer 1, Sheet 1 that

$$S^3 = \underbrace{\{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1,\ |z|^2 \geq \tfrac{1}{2}\}}_{=: H_1} \cup \underbrace{\{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1,\ |w|^2 \geq \tfrac{1}{2}\}}_{=: H_2}$$

is a Heegaard splitting of S^3 of genus 1.

Indeed, $H_1 \xrightarrow{\cong_{C^\infty}} S^1 \times D^2 = \{(z,w) \in \mathbb{C}^2 \mid |z| = 1,\ |w| \leq 1\}$

via $(z,w) \longmapsto \left(\frac{z}{|z|}, \frac{w}{|w|}\right)$ $\left(\text{Inverse: } (a,b) \mapsto \left(\frac{a}{\sqrt{|a|^2+|b|^2}}, \frac{b}{\sqrt{|a|^2+|b|^2}}\right)\right)$

and similarly $H_2 \xrightarrow{\cong_{C^\infty}} D^2 \times S^1$ $\left((z,w) \mapsto \left(\frac{z}{|w|}, \frac{w}{|w|}\right);\ \text{same inverse}\right)$

The action of $\mathbb{Z}/p\mathbb{Z}$ on S^3 (defining $L_{p,q}$) preserves H_1 & H_2 setwise (b/c $|e^{2\pi i/p} z| = |e^{2\pi i/p}|\,|z| = |z|$, $|e^{2\pi i q/p} w| = |w|$ using $|e^{ix}| = 1, x \in \mathbb{R}$).

One can show that

$$H_1/\mathbb{Z}/p\mathbb{Z} \xrightarrow{\cong_{C^\infty}} S^1 \times D^2, \quad [(z,w)] \mapsto \left(\frac{z^p}{|z^p|}, \frac{z^{-q} w}{|z^{1-q}|}\right)$$

and $$H_2/\mathbb{Z}/p\mathbb{Z} \xrightarrow{\cong_{C^\infty}} D^2 \times S^1, \quad [(z,w)] \mapsto \left(\frac{z w^r}{|w^{1+r}|}, \frac{w^p}{|w^p|}\right)$$

where $r \cdot q \equiv -1 \mod p$

are well-defined diffeomorphisms. [Friedl, Lecture notes for Alg. Topology I–IV, p. 4073, Section 215.2 "Heegaard splittings of 3-dim. smooth mfds]

$$\Rightarrow \quad H_1/\mathbb{Z}/p\mathbb{Z} \cup H_2/\mathbb{Z}/p\mathbb{Z} \cong_{C^\infty} L_{p,q} \text{ is a Heegaard}$$

splitting of genus 1. $\square$

How to see this geometrically:

Under the diffeos $H_1 \cong_{C^\infty} S^1 \times D^2$, $H_2 \cong_{C^\infty} D^2 \times S^1$,

the action of $\mathbb{Z}/p\mathbb{Z}$ descends to

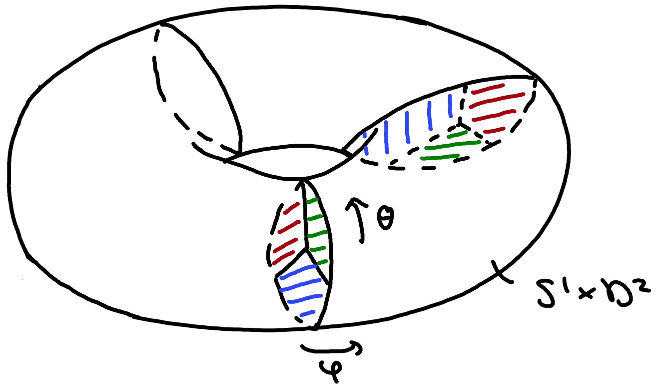
$\mathbb{Z}/p\mathbb{Z} \curvearrowright S^1 \times D^2 = \{(a,b) \in \mathbb{C}^2 \mid |a| = 1, |b| \le 1\}$

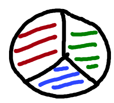
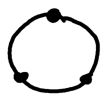
$$1 \cdot (a,b) = (e^{2\pi i/p} a, e^{2\pi i q/p} b).$$

In polar coordinates, $(a,b) = (e^{i\varphi}, re^{i\theta}) \in S^1 \times D^2$,

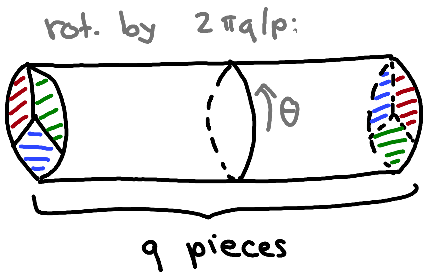
$$1 \cdot (a,b) = \left(e^{i(\varphi + 2\pi/p)}, re^{i(\theta + 2\pi q/p)}\right).$$

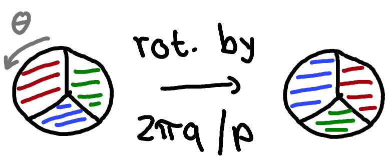
Ex: $p = 3, q = 2$

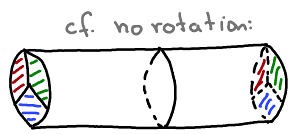


- Divide D^2-factor into p equal/congruent "pizza slices" 
- Divide S^1-factor into p equal/congruent pieces 
- Now, on the S^1-factor, the action by $1 \in \mathbb{Z}/p\mathbb{Z}$ moves each of these pieces "1 step" b/c $\frac{2\pi/p}{2\pi} = \frac{1}{p}$
- On the D^2-factor: "q steps" b/c $\frac{2\pi q/p}{2\pi} = \frac{q}{p} = q \cdot \frac{1}{p}$

Consider a fundamental region for the action: (w/ identifications on the bdry)





[

]

On $H_2 \cong D^2 \times S^1$, the action moves "1 step" on the D^2-factor "q steps" on the S^1-factor.

With the terms "meridian" and "longitude" defined below, one can make this geometric part more precise.

Towards "$\Rightarrow$":

Def: Let V be an oriented solid torus, i.e.

$$V \underset{\substack{\uparrow \\ \text{or. preserving}}}{\cong_{C^\infty}} S^1 \times D^2.$$

A meridian μ of V is a simple closed curve on ∂V

(closed curve: continuous map $S^1 \to \partial V$
simple: injective
s.c.c. $\rightsquigarrow$ (top.) emb. onto its image)

s.t. μ is homotopically trivial in V, $\circledast$ but

homotopically nontrivial in ∂V. [$\rightsquigarrow$ essential in ∂V]

[$\circledast \overset{\text{Rolfsen,}}{\underset{2.E.1}{\Longleftrightarrow}}$ μ bounds a disk in V]

Ex:

Fact: (Rolfsen, 2.E) • Meridians are unique up to isotopy (and up to ambient isotopy of V).

• $\forall$ meridians μ, there exists an (or. pres.) diffeo $h: S^1 \times D^2 \to V$ (a framing of V) s.t. $h(\{1\} \times \partial D^2) = \mu$.

Def (cont'd): A longitude λ of V is $h(S^1 \times \{1\})$ for some framing h of V.

[$\overset{\text{Rolfsen}}{\underset{2.E}{\Longleftrightarrow}}$ λ represents a generator of $\pi_1(V)$ $(\cong H_1(V)) \cong \mathbb{Z}$

$\Longleftrightarrow$ λ intersects some meridian μ of V transversally in a single pt]

Orient μ, λ s.t. (μ, λ) represent positive orientation of ∂V

Rem: Longitudes are <u>not</u> unique.

$\lambda' = \lambda + n\mu$, $n \in \mathbb{Z}$
all longitudes

Lemma: (1) Every simple closed curve on $\partial V \cong T^2$ is isotopic to exactly one curve of the form $m\mu + n\lambda$ for m, n coprime integers.

(2) $\rightsquigarrow$ Lecture 7.

Pf idea for part (1): [Rolfsen, Chapter 2, in part. Thm 2.C.2
Farb-Margalit, Prop. 1.5; see also Prop. 1.10]
$\downarrow$
"A primer on mapping class groups"

Consider the universal cover $\mathbb{R}^2 \to T^2 \cong_{C^\infty} \mathbb{R}^2/\mathbb{Z}^2$:
$(m,n) \in \mathbb{Z}^2 \cong \pi_1(T^2)$ is represented by proj. of straight line in $\mathbb{R}^2$ from $(0,0)$ to (m,n).
This curve is simple iff $\gcd(m,n) = 1$.

e.g. $m=2$, $n=3$