

Lecture 6

- Oral exam dates:
- Feb 10-14, 2025
 - March 17-19, 2025

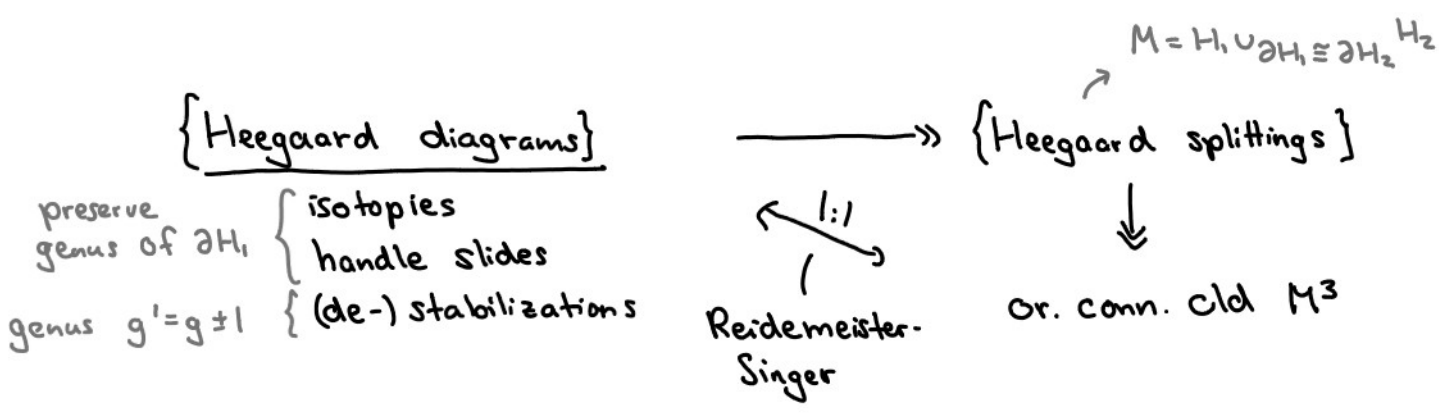
Please let me know ahead of time (end of January) if you have special needs for the exam or have time conflicts with other exams on certain dates/ at certain times. We'll schedule the precise times at the end of the semester.

- Next week:
- Aru Ray
 - evaluation

Dec 4: No lecture (Dies academicus)

Last two lectures:

- §. 3 Heegaard splittings
- §. 3.1 Existence
- §. 3.2 Heegaard diagrams



Today: § 3.3 Heegaard genus, lens spaces

§ 3.3 Heegaard genus

Def: The Heegaard genus of an (or., closed, conn.) M^3 is

$$g(M) := \min_{\substack{\uparrow \\ [0 \leq g(M) < \infty]}} \{g(\Sigma) \mid M^3 = H_1 \cup_{\Sigma} H_2 \text{ Heegaard splitting of } M\}.$$

\uparrow Heegaard surface of the Heegaard splitting

Prop: $g(M^3) = 0 \Leftrightarrow M \cong_{C^0} S^3$. ($\Leftrightarrow M \cong_{C^\infty} S^3$)
 [Bing, Moise, Munkres; see Lect 2.]

Pf: " \Leftarrow ": \checkmark ($S^3 = D^3 \cup_{S^2} D^3$)

" \Rightarrow ": Use Exer 4c, Exercise Sheet 2. □

Cor: $g(S^1 \times S^2) = 1$.

Pf: We saw " \leq ": $S^1 \times S^2 \cong \text{[diagram of } S^1 \times S^2 \text{ as } S^1 \times D^2 \cup_{\varphi} S^1 \times D^2 \text{]} = \text{[diagram of } S^1 \times S^2 \text{ as } D^2 \cup_{\varphi} D^2 \text{]}.$
 [= $S^1 \times D^2 \cup_{\varphi} S^1 \times D^2$, note that $D^2 \cup_{\varphi} D^2 = S^2$.]

" \geq " follows from the prop. using e.g. $\pi_1(S^1 \times S^2) \cong \mathbb{Z} \neq \{1\} = \pi_1(S^3)$.

[Suppose $g(S^1 \times S^2) = 0 \xrightarrow{\text{Prop}} S^1 \times S^2 \cong_{C^0} S^3 \begin{cases} \text{to } \uparrow \\ \text{b/c homeomorphic mfd's have} \\ \text{isomorphic } \pi_1 \end{cases}$
 ($X \cong_{C^0} Y \Rightarrow X \cong_{\text{h.e.}} Y \Rightarrow \pi_1(X) \cong \pi_1(Y)$)]

Recall:

Lens space $L_{p,q} := S^3 / \sim$ where

[studied by Tietze in 1908]

$S^3 = \{ (z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1 \}$, $p, q \in \mathbb{Z}$ coprime, $p \neq 0$,
 ($\gcd(p, q) = 1$)

and $[m] \circ (z, w) := (e^{2\pi i m/p} z, e^{2\pi i q m/p} w)$ for $[m] \in \mathbb{Z}/p\mathbb{Z}$.

(action of $\mathbb{Z}/p\mathbb{Z}$ on S^3 generated by

$$(z, w) \sim 1 \circ (z, w) = (e^{2\pi i/p} z, e^{2\pi i q/p} w).$$

(See Lect. 1 & Exer Sheet 1, Exer 2.)

↳ this action is free, smooth & proper

→ quotient $L_{p,q}$ is (smooth) 3-mfd

Ex:

• $L_{1,0} \cong_{C^\infty} S^3$, $L_{2,1} \cong_{C^\infty} \mathbb{R}P^3$ (see Exer 2).

($\cong S^3 / (z, w) \sim (-z, -w)$)

• $L_{1,q} \cong_{C^\infty} S^3 \forall q$ (exer).

• Indeed, $L_{p,q+kp} \cong L_{p,q} \forall k \in \mathbb{Z}$ (b/c $e^{2\pi i k} = 1$.)

Thm 1: $g(M^3)=1 \Leftrightarrow M$ is homeomorphic to a lens space
 $L_{p,q} \cong_{\mathbb{C}} S^3$ or $S^1 \times S^2 =: L_{0,1}$.

A little more context before we prove this theorem:

- Mfds M^3 w/ $g(M^3) \geq 2$ are much less well-understood

Ex: $g(T^3)=3$: Using Exer 3, Sheet 2 (hard?), you can show
 \parallel
 $S^1 \times S^1 \times S^1$ that $g(T^3) \leq 3$.

Exer: Use π_1 (or H_1) to show " \geq ".
 (Exercise Sheet 3)

• Why are lens spaces cool?

Fact / Exer: From Sheet 1, we know $\pi_1(L_{p,q}) \cong \mathbb{Z}/p\mathbb{Z}$.

Indeed, the homotopy groups π_k and
 homology groups H_k of $L_{p,q}$ only depend on p !

⌈ • $\pi_k(L_{p,q})$ indep. of q : $S^3 \rightarrow L_{p,q} = S^3/\mathbb{Z}/p$ is the
 universal covering map, LES (long exact sequence) of
 (\leadsto fibration)
 homotopy groups $\Rightarrow \pi_k(L_{p,q}) \cong \pi_k(S^3) \forall k \geq 2$.

• Fact / exer: π_1 determines $H_k \forall k$ for any closed, conn., or. M^3
 (Sheet 3) $\Rightarrow H_k(L_{p,q}) \cong \begin{cases} \mathbb{Z}, & k=0, 3 \\ \mathbb{Z}/p\mathbb{Z}, & k=1 \\ \{0\}, & k=2, k \geq 4 \end{cases}$]

$\Rightarrow L_{p,q} \stackrel{\cong}{\sim} L_{p',q'}$ only possible if $p=p'$
 \uparrow
 homotopy equivalence

In fact, there are the following classifications:

Thm 2 (Whitehead 1941, Reidemeister 1935, Brody 1960):

↳ homotopy equiv.

↳ PL homeo

↳ homeo

- $L_{p,q} \underset{\substack{\cong \\ \uparrow \\ \text{homotopy} \\ \text{equivalence}}}{\cong} L_{p',q'} \iff p=p' \text{ and } \pm qq' = m^2 \pmod{p}$
for some $m \in \mathbb{N}$
($\pm qq'$ quadratic residue mod p)

[This can be shown using the "torsion linking pairing"]

- $L_{p,q} \underset{\substack{\cong_{C^0} \\ \text{or } \cong_{C^\infty} \\ \text{(equiv.!)}}}{\cong} L_{p',q'} \iff p=p' \text{ and } q' = \pm q^{\pm 1} \pmod{p}$
[" \Leftarrow " Exer.
" \Rightarrow " e.g. using "Reidemeister torsion";
see e.g. book by V. Turaev,
"Torsions of 3-dimensional manifolds"]

Ex (Cor. of this Thm) $L_{7,1} \cong L_{7,2}$ (b/c $1 \cdot 2 = 3^2 \pmod{7}$),
but $L_{7,1} \not\cong_{C^0} L_{7,2}$ (b/c $2^{-1} = 4 \pmod{7}$).

↳ "geometric topology of lens spaces differs from their algebraic topology."

[Other ex: $L_{5,1} \not\cong_{C^0} L_{5,2}$ [$2^{-1} = 3 \pmod{5}$]

Studied by Alexander in 1919 and historically first ex. of 3-mfds

M_1, M_2 w/ isom. π_1 & homology groups, but $M_1 \not\cong_{C^0} M_2$

(Now we know that $L_{5,1} \neq L_{5,2}$ which of course implies $L_{5,1} \not\cong_{C^0} L_{5,2}$.)

Rem/ "fun fact": Using number theory, one can determine

how many different homotopy types there are among

$L_{p,1}, L_{p,2}, \dots, L_{p,p-1}$ for any $p > 0$ (if p is not a prime, consider only $L_{p,q}$ w/ $\gcd(p,q)=1$).

Ex: [See Rolfsen, Rem. 9.B.7.] For prime p , there

- are precisely two homotopy types if $p \equiv 1 \pmod{4}$
- is exactly one " " type if $p \equiv 3 \pmod{4}$.

Pf of " \Leftarrow " of Thm 1:

We need to show that $g(L_{p,q}) = 1$ for $p \neq 1$.

Recall from Exer 1, Sheet 1 that

$$S^3 = \underbrace{\left\{ (z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1, |z|^2 \geq \frac{1}{2} \right\}}_{=: H_1} \cup \underbrace{\left\{ (z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1, |w|^2 \geq \frac{1}{2} \right\}}_{=: H_2}$$

is a Heegaard splitting of S^3 of genus 1.

Indeed, $H_1 \xrightarrow{\cong_{C^\infty}} S^1 \times D^2 = \{ (z,w) \in \mathbb{C}^2 \mid |z|=1, |w| \leq 1 \}$

$$\text{via } (z,w) \mapsto \left(\frac{z}{|z|}, \frac{w}{|z|} \right) \quad \left(\text{Inverse: } (a,b) \mapsto \left(\frac{a}{\sqrt{|a|^2 + |b|^2}}, \frac{b}{\sqrt{|a|^2 + |b|^2}} \right) \right)$$

and similarly $H_2 \xrightarrow{\cong_{C^\infty}} D^2 \times S^1 \quad \left((z,w) \mapsto \left(\frac{z}{|w|}, \frac{w}{|w|} \right); \text{ same in verse} \right)$

The action of $\mathbb{Z}/p\mathbb{Z}$ on S^3 (defining $L_{p,q}$) preserves H_1 & H_2 setwise
(bc $|e^{2\pi i/p} z| = |e^{2\pi i/p}| |z| = |z|$, $|e^{2\pi i q/p} w| = |w|$ using $|e^{ix}| = 1, x \in \mathbb{R}$).

One can show that

$$H_1 / \mathbb{Z}/p\mathbb{Z} \xrightarrow{\cong_{C^\infty}} S^1 \times D^2, \quad [(z,w)] \mapsto \left(\frac{z^p}{|z|^p}, \frac{z^{-q} w}{|z|^{1-q}} \right)$$

$$\text{and } H_2 / \mathbb{Z}/p\mathbb{Z} \xrightarrow{\cong_{C^\infty}} D^2 \times S^1, \quad [(z,w)] \mapsto \left(\frac{z w^r}{|w|^{1+r}}, \frac{w^p}{|w|^p} \right)$$

where $r \cdot q \equiv -1 \pmod{p}$

are well-defined diffeomorphisms. [Friedl, Lecture notes for Alg. Topology I-IV, p. 4073, Section 215.2 "Heegaard splittings of 3-dim. smooth mfd's"]

$\Rightarrow H_1 / \mathbb{Z}/p\mathbb{Z} \cup H_2 / \mathbb{Z}/p\mathbb{Z} \cong_{C^\infty} L_{p,q}$ is a Heegaard splitting of genus 1. □

How to see this geometrically:

Under the diffeos $H_1 \cong_{C^\infty} S^1 \times D^2$, $H_2 \cong_{C^\infty} D^2 \times S^1$,

the action of $\mathbb{Z}/p\mathbb{Z}$ descends to

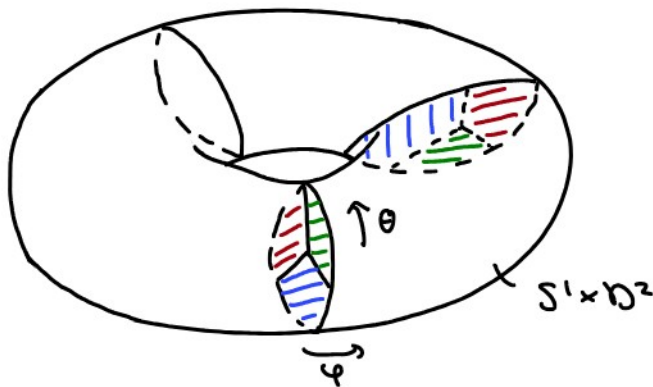
$$\mathbb{Z}/p\mathbb{Z} \curvearrowright S^1 \times D^2 = \{(a,b) \in \mathbb{C}^2 \mid |a|=1, |b| \leq 1\}$$



$$l \cdot (a,b) = (e^{2\pi i l/p} a, e^{2\pi i q l/p} b).$$

In polar coordinates, $(a,b) = (e^{i\varphi}, r e^{i\theta}) \in S^1 \times D^2$,

$$l \cdot (a,b) = (e^{i(\varphi + 2\pi l/p)}, r e^{i(\theta + 2\pi q l/p)}).$$

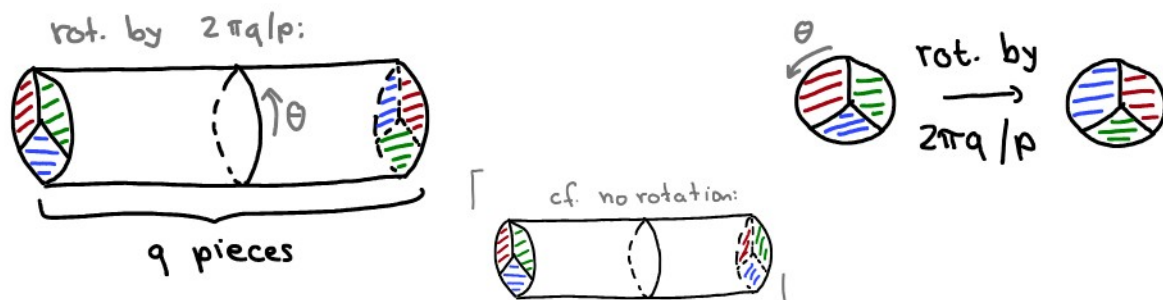
Ex: $p=3, q=2$



- Divide D^2 -factor into p equal/congruent "pizza slices" 
- Divide S^1 -factor into p equal/congruent pieces 
- Now, on the S^1 -factor, the action by $l \in \mathbb{Z}/p\mathbb{Z}$ moves each of these pieces "1 step" b/c $\frac{2\pi l/p}{2\pi} = \frac{l}{p}$
- On the D^2 -factor: "q steps" b/c $\frac{2\pi q l/p}{2\pi} = \frac{q}{p} = \underline{\underline{q \cdot \frac{1}{p}}}$

Consider a fundamental region

for the action: (w/ identifications on the bdry)



On $H_2 \cong D^2 \times S^1$, the action moves "1 step" on the D^2 -factor
"q steps" on the S^1 -factor.

With the terms "meridian" and "longitude" defined below,
one can make this geometric part more precise.

Towards "=>":

Def: Let V be an oriented solid torus, i.e.

$$V \cong_{\substack{\uparrow \\ \text{or. preserving}}} C^\infty S^1 \times D^2.$$

A meridian μ of V is a simple closed curve on ∂V
(closed curve: continuous map $S^1 \rightarrow \partial V$
simple: injective
s.c.c. \rightarrow (top.) emb. onto its image)

s.t. μ is homotopically trivial in V ,[⊕] but
homotopically non trivial in ∂V . [\rightarrow essential in ∂V]

Ex:



[⊕ Rolfsen,
2.E.1 μ bounds a disk in V]

Fact: (Rolfsen, 2.E) • Meridians are unique up to isotopy
(and up to ambient isotopy of V).

- \forall meridians μ , there exists an (or. pres.)
diffeo $h: S^1 \times D^2 \rightarrow V$ (a framing of V)
s.t. $h(\{1\} \times \partial D^2) = \mu$.

Def. (cont'd): A longitude λ of V is $h(S^1 \times \{1\})$ for some framing
 h of V .

[⊕ Rolfsen
2.E λ represents a generator of $\pi_1(V) (\cong H_1(V)) \cong \mathbb{Z}$
 λ intersects some meridian μ of V transversally
in a single pt]

Orient μ, λ s.t. (μ, λ) represents positive orientation of ∂V



Rem: Longitudes are not unique.



$$\lambda' = \lambda + n\mu, n \in \mathbb{Z}$$

all longitudes

Lemma: (1) Every simple closed curve on $\partial V \cong T^2$ is isotopic to exactly one curve of the form $m\mu + n\lambda$ for m, n coprime integers.
(2) \leadsto Lecture 7.

Pf idea for part (1): [Rolfsen, Chapter 2, in part. Thm 2.C.2
Farb-Margalit, Prop. 1.5; see also Prop. 1.10]
 \downarrow
"A primer on mapping class groups"

Consider the universal cover $\mathbb{R}^2 \rightarrow T^2 \cong_{\text{C}} \mathbb{R}^2 / \mathbb{Z}^2$:

$(m, n) \in \mathbb{Z}^2 \cong \pi_1(T^2)$ is represented by proj. of
straight line in \mathbb{R}^2 from $(0, 0)$ to (m, n) .

This curve is simple iff $\gcd(m, n) = 1$.

