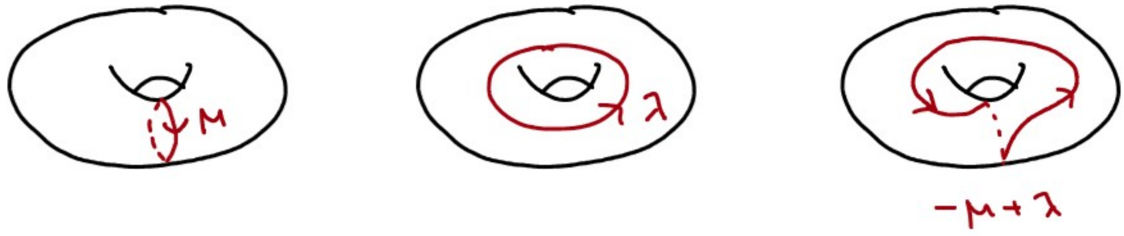
Lecture 7 (given by Aru Ray)

Last time: Heegaard genus, $L_{p,q}$ has genus one.

Today:
- Heegaard genus one $\Rightarrow$ $L_{p,q}$ or $L_{0,1} := S^1 \times S^2$
- course evaluation
- Knots and links

Next week: Dies Academicus

Lemma: (1) Every simple closed curve on $\partial V \cong T^2$ is isotopic to exactly one curve of the form $m\mu + n\lambda$ for m, n coprime integers. } last week (Lecture 6)



(signs!)

(2) $\underbrace{\mathrm{Homeo}(T^2)/\text{isotopy}}_{\text{[extended] mapping class group}} \xrightarrow[\text{iso}]{\cong} \mathrm{Aut}(\underbrace{\pi_1(T^2)}_{\cong \mathbb{Z}^2}) \overset{\text{choose basis } \{\mu,\lambda\}}{\cong} GL_2(\mathbb{Z}) := \{A \in M_{2\times 2}(\mathbb{Z}) \mid \det(A) = \pm 1\}$

(isomorphism)

$h \longmapsto \pi_1 h = h_* : \pi_1(T^2) \cong \mathbb{Z}^2_{\langle \mu,\lambda\rangle} \to \pi_1(T^2)$

Note: In general this map would have image the outer automorphism group! (but $\pi_1(T^2) \cong \mathbb{Z}^2$ is abelian)

[Dehn-Nielsen-Baer Thm see Thm 8.1 in Farb-Margalit]

[$\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ for a gp G where the inner automorphisms are conjugations.]

Note: This sort of behavior ($MCG \cong \mathrm{Out}(\pi_1)$) holds more generally for Eilenberg-MacLane spaces.

Pf of (2): • Rolfsen, 2.D.4

• see also Farb-Margalit, Thm 2.5

Surjectivity: Realize $\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. These generate $GL_2(\mathbb{Z})$.

Injectivity: NTS $f \in \ker(..) \Leftrightarrow f$ iso to id. [Rolfsen, 2.D.3]

"$\Leftarrow$": easy

"$\Rightarrow$": Step by step "straigthening"

$h_* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $h(\mu) = \mu$. Make h id on nbhd of μ, then tub. nbhd of λ, then Alexander trick. □

Back to lens spaces.

We showed $S^3/_{\mathbb{Z}/p\mathbb{Z}} =: L_{p,q} \cong \overbrace{S^1 \times D^2}^{V_1} \cup_\varphi \overbrace{S^1 \times D^2}^{V_2}$ ($\cong D^2 \times S^1$)

$\qquad\qquad\qquad\qquad\qquad\qquad (S^1 \times D^2)/_{\mathbb{Z}/p\mathbb{Z}} \cong S^1 \times D^2, \quad (S^1 \times D^2)/_{\mathbb{Z}/p\mathbb{Z}} \cong S^1 \times D^2$

where $\varphi : \underset{\partial V_2}{\partial(S^1 \times D^2)} \cong T^2 \to \underset{\partial V_1}{\partial(S^1 \times D^2)} \cong T^2$.

Pf from last time (Lecture 6) shows:

$$\varphi(\mu_2) = p\lambda_1 + q\mu_1$$

$$\varphi(\lambda_2) = r\mu_1 + s\lambda_1$$

so $\varphi_* : \pi_1(T^2) \circlearrowleft$ given by

$$(\mu_2, \lambda_2) \mapsto (q\mu_1 + p\lambda_1, r\mu_1 + s\lambda_1) = \begin{pmatrix} q & p \\ r & s \end{pmatrix} \begin{pmatrix} \mu_1 \\ \lambda_1 \end{pmatrix}$$

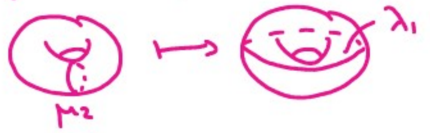
← corrected version

"Sanity check:"

$L_{1,0} \cong_{C^\infty} S^3$:

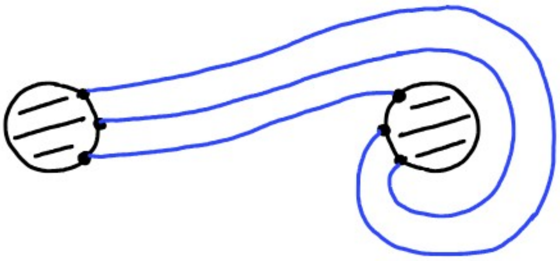
$p = 1, q = 0 \quad \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ w/ det $= -1$ (or. rev.)

$\mu_2 \overset{\varphi}{\mapsto} \lambda_1$

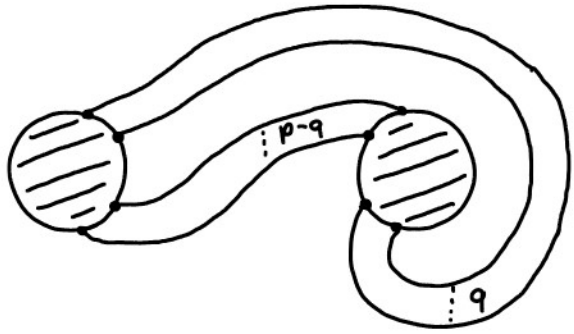


Ex: $p = 3$, $q = 2$

Heegaard diagram:



For general p, q:



(draw p pts, then connect)

Sketch of pf of "$\Rightarrow$" of Thm 1

see Lecture 6

Let M^3 be smooth, or., cld, conn. and suppose that M has a genus one Heegaard splitting. NTS: $M \cong L_{p,q}$ for some p, q

(including $L_{1,0} \cong S^3$, $L_{0,1} = S^1 \times S^2$).

$M = H_1 \cup_\varphi H_2$, where $H_i \cong_{C^\infty} S^1 \times D^2$ and

$$\varphi : \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ T^2}}{\partial H_2} \to \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ T^2}}{\partial H_1}$$ or. reversing diffeo.

(cf. Thm 1 in Lecture 4)

By the Lemma above (part (2)), φ is isotopic to $\varphi' : T^2 \to T^2$,

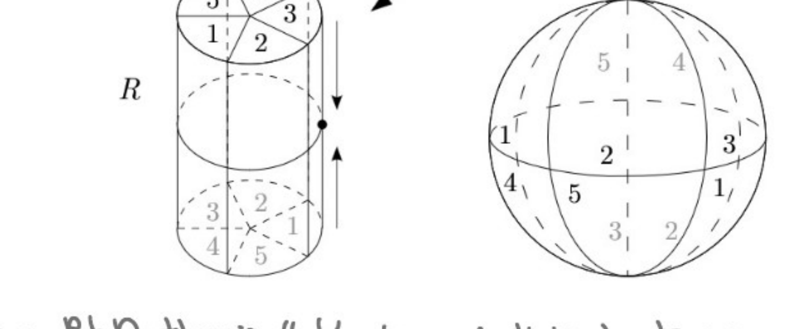
[note that isotopic φ, φ' give diffeomorphic mfds]

$$\mu_2 \mapsto p\mu_1 + q\lambda_1,$$

$$\lambda_2 \mapsto r\mu_1 + s\lambda_1 \quad \text{w/} \quad \det\begin{pmatrix} p & q \\ r & s \end{pmatrix} = -1. \qquad \square$$

or. rev.

Another ex, $p = 5, q = 2$:



$L_{p,q}$ can also be viewed as quotient of the ball D^3; this is what the picture on the bottom right is referring to. [See also Rolfsen, 9.B.8.]

Source: PhD thesis "Knots and links in lens spaces" by Enrico Manfredi (Bologna 2014), p. 13

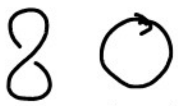
§ 4. Dehn surgery

§ 4.1 Knots and links (needed to define Dehn surgery)

Def: A <u>link</u> of m components is a smooth embedding
$$L: \bigsqcup_m S^1 \hookrightarrow S^3.$$

A <u>knot</u> is a link w/ $m=1$. ($\rightsquigarrow$ connected; emb. $S^1 \hookrightarrow S^3$)

Ex:



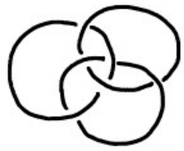
unknot

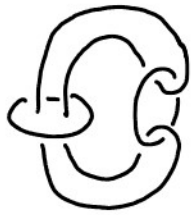
trefoil

fig. 8

unlink

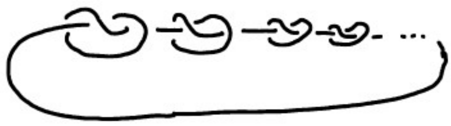
Hopf link





Borromean rings

Rem: $\exists$ "wild" knots:



not <u>smooth</u>!

Note: Some consider <u>maps</u>, others <u>images</u>

some <u>oriented</u>, others <u>unoriented</u>

some <u>ordered</u>, others not (rarely).

Def: Two links L_0, L_1 are isotopic if $\exists$ smooth homotopy
$$h_t: \bigsqcup_m S^1 \hookrightarrow S^3 \text{ w/ } h_0 = L_0,\ h_1 = L_1,\ h_t \text{ smooth emb. } \forall t.$$

Rem: By isotopy extension theorem (see Lecture 2|3),

L_0, L_1 isotopic $\Rightarrow$ L_0, L_1 ambient isotopic,
(through h_t) i.e. $\exists\ H: S^3 \times I \to S^3$ s.t. $H_0 = id$, H_t diffeo $\forall t$.
($\rightsquigarrow h_t = H_t \circ h_0$)

In particular, $\exists\ H_1: S^3 \to S^3$ or. pres. diffeo w/ $H_1 \circ L_0 = L_1$.

[n.b. Since $\mathrm{Diffeo}^+(S^3)/_{iso} =: MCG(S^3) = 1$, that is also equivalent to L_0, L_1 being isotopic.]

Rem: from now on, a "knot" or "link" refers to its isotopy class (if not otherwise specified).

How to tell knots / links apart? ⇝ Need an invariant

Def (linking number): Let K, J be disjoint, oriented knots in S^3.

Fact: $H_1(S^3 \setminus K) \cong \mathbb{Z} \langle \mu_K \rangle$ (Exer, Sheet 4)

meridian (oriented)

$\operatorname{lk}(K, J) := m$ where $[J] = m \in H_1(S^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z}$.

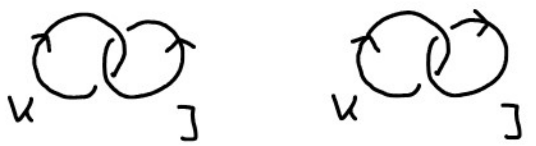
⚠ $H_1(S^3 \setminus K) \cong \pi_1(S^3 \setminus K)_{ab}$, the abelianization of $\pi_1(S^3 \setminus K)$ (abel. of a gp π is $\pi / [\pi, \pi]$ where $[\pi, \pi]$ is the commutator subgp)

If you are not so familiar w/ homology, think of $\pi_1(S^3 \setminus K)_{ab}$.

Or use homology-free, diagrammatic def'n below.

Check: This is an isotopy invariant, i.e. $K = K'$ (isotopic), $J = J' \Rightarrow \operatorname{lk}(K, J) = \operatorname{lk}(K', J')$.

Ex:



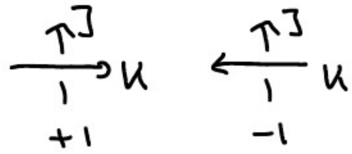
negative Hopf link — $\operatorname{lk}(K, J) = -1$

positive Hopf link — $\operatorname{lk}(K, J) = +1$

Rem: There are lots of equivalent definitions of the linking number, some better for computations, some better for proofs.

See e.g. 5.D in Rolfsen.

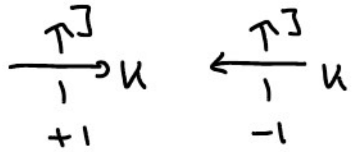
How to <u>compute</u> the linking nr?



Draw a link diagram (image under projection $L \subset \underset{\mathbb{R}^3 \cup \{\infty\}}{\underset{\wr\wr}{S^3}} \to \mathbb{R}^2$

w/out triple pts; draw crossings w/ "gaps")

Assign signs to <u>crossings</u> (double pts of the projection) where J passes below/under K:



$$\rightsquigarrow \ lk(K,J) = \sum_{\substack{\text{cross.} \\ J \text{ under } K}} \pm 1$$

<u>Ex:</u>

$lk(K,J) = +2$