Lecture 8

Last time: started § 4. Dehn surgery
§ 4.1 Knots and links

Links $L: \coprod_{i=1}^{m} S^1 \hookrightarrow S^3$

Knot : $m=1$

linking number $lk(K,J) := [J] \in \mathbb{Z} \cong H_1(S^3 \setminus K) = \pi_1(S^3\setminus K)_{ab}$ ← Exer Sheet 4
$\cong \mathbb{Z}\langle \mu_K \rangle$ ← def'd later today

of oriented knots K, J
(isotopy invariant of knots)
(first studied by Gauss)

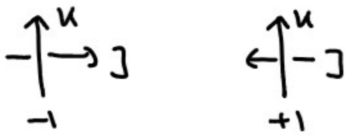
Ex:

oriented link $K \cup J$

$lk(K,J) = +1$

J isotopic to μ_K ← oriented by right-hand rule

How to compute:



$\rightsquigarrow lk(K,J) = \sum_{\substack{\text{crossings} \\ J \text{ under } K}} \pm 1$

Def: Let K be an oriented knot.

A (closed) <u>tubular neighborhood</u> of K is a smooth embedding $t: S^1 \times D^2 \hookrightarrow S^3$ s.t. $t|_{S^1\times\{0\}} = K$

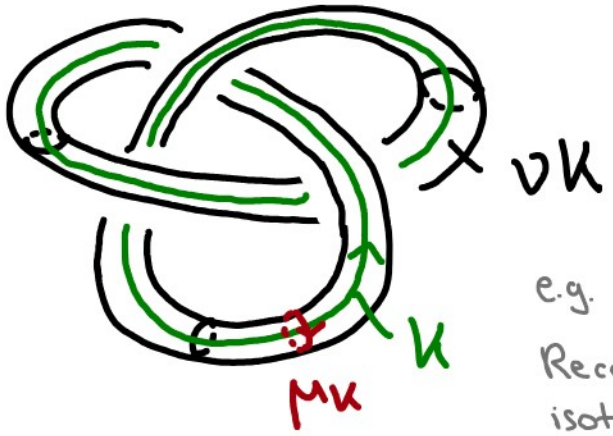
(i.e. $t(x,0) = K(x)$).

[disk bundle over $K(S^1) \subset S^3$ ...]

Write $\nu K := t(S^1 \times D^2) \subseteq S^3$. ($\leadsto$ smooth submfd of S, $\cong_{C^\infty} S^1 \times D^2$)

Rem: Tubular neighborhoods exist and are unique (up to ambient isotopy fixing K); see e.g. Wall, Thms 2.3.3 & 2.5.5.

A <u>meridian</u> μ_K of K is a meridian of νK, i.e. a s.c.c. on $\partial \nu K \cong S^1 \times S^1$ s.t. $\mu_K \simeq \{pt\}$ in νK, $\mu_K \not\simeq \{pt\}$ in $\partial\nu K$.

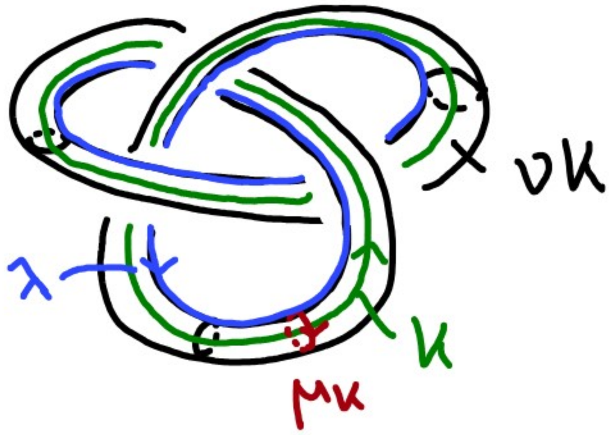


e.g. $\mu_K := t(\{1\} \times \partial D^2)$ is a meridian.

Recall that meridians are unique up to isotopy.

Convention: Orient μ_K s.t. $lk(\mu_K, K) = +1$.

A <u>longitude</u> λ of K is a longitude of νK, e.g. $t(S^1 \times \{1\})$.

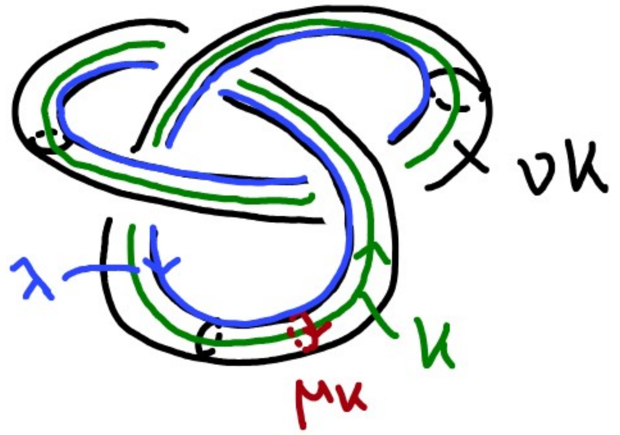


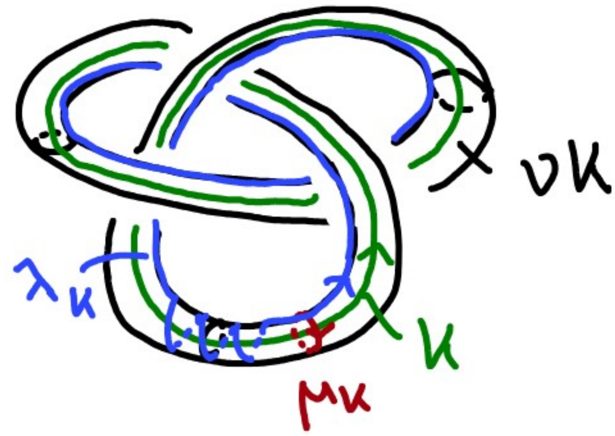
Recall that longitudes are <u>not</u> unique.

The <u>Seifert longitude</u> λ_K of K is the (isotopy class of the) longitude of K s.t. $lk(\lambda_K, K) = 0$, where we choose the "same" orientation on λ_K as on K.

<u>Rem:</u> • Meridians generate the kernel of $H_1(\partial\nu K) \to H_1(\nu K)$.
$$H_1(\partial\nu K) \cong \mathbb{Z}^2, \quad H_1(\nu K) \cong \mathbb{Z}$$

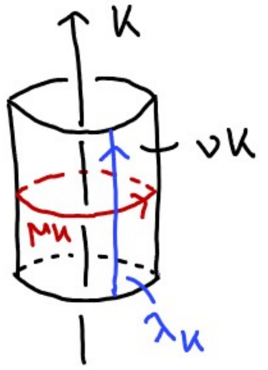
• λ_K is the longitude which generates the kernel of $H_1(\partial\nu K) \to H_1(S^3\setminus\nu K)$ [unique up to amb. isotopy]





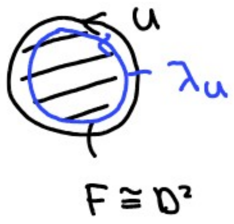
$lk(\lambda, K) = -3$ $\qquad\qquad$ $lk(\lambda_K, K) = 0$

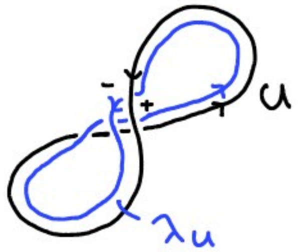
["blackboard framing" $\rightsquigarrow$ $lk(K$, "blackboard push-off"$)$
$=$ writhe $(K) :=$ signed nr of self-crossings of K]



<u>Rem:</u> $\forall$ oriented knots K $\exists$ <u>Seifert surface</u> for K, i.e. an or., conn., cpct surface (smooth emb. of a surface) F w/ oriented boundary $\partial F = K$ (see Exer Sheet 4). (bonus exer)
One can find λ_K by pushing K into a Seifert surface for K. ($\rightsquigarrow$ obtain λ_K e.g. from inward pointing normal to Seifert surface)

Ex:





Fact / exer: The Seifert longitude is well def'd, i.e. there is a unique choice.

Pf idea: Given μ_K and a choice of λ_K, by Lemma from Lecture 6/7 any longitude of νK is of the form $a\mu_K + b\lambda_K$. Using $lk(\lambda_K, K) = 0$ & $lk(\mu_K, K) = +1$, we obtain $lk(a\mu_K + b\lambda_K, K) = a$. □

[Note also that $lk(\lambda_K, K) = 0$ is independent of the orientation of K; $lk(\overleftarrow{K}, J) = lk(K, \overleftarrow{J}) = -lk(K, J)$.]

↑ K w/ reverse orientation

§ 4.2 Dehn surgery: definition & main thm

Def: Let p, q be coprime integers. The manifold obtained by Dehn surgery along K w/ surgery coefficient (or slope) $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ is defined as

$$K_{p/q} := S^3_K\left(\frac{p}{q}\right) := \overbrace{S^3 \setminus \mathrm{Int}(\nu K)}^{\text{Knot exterior}} \cup_\varphi S^1 \times D^2$$

(enough to find homeo (C^0))

where $\varphi: \partial(S^1 \times D^2) \cong S^1 \times S^1 \xrightarrow{\cong_{C^\infty}} \partial(S^3 \setminus \mathrm{Int}(\nu K)) \cong \partial \nu K \cong S^1 \times S^1$,

$$\{1\} \times \partial D^2 \longmapsto p\mu_K + q\lambda_K.$$

↑ or any other pt $\in S^1$

Recall: any s.c.c. on $\partial \nu K$ is given by $p\mu_K + q\lambda_K$ for some coprime p, q. [Lemma in Lect 6/7].

Lemma: 1) $K_{p/q}$ only depends on the quotient $\frac{p}{q}$

Pf: $\frac{-p}{-q} = \frac{p}{q}$ this sign change corresponds to changing the orientation of both μ_K & λ_K

$\leftrightarrow$ changing orient. of K or orient. of $\varphi(\{1\} \times \partial D^2)$.

$\rightsquigarrow$ diffeom. mfds [PS, p. 104 or GS, p. 157]

2) $K_{p/q}$ is independent of choice of φ (up to diffeo)

Sketch: $K_{p/q} = S^3 \setminus \mathrm{Int}(\nu K) \cup_\varphi S^1 \times D^2$

$S^1 \times D^2 = h_0 \cup h_1 \cong h_3 \cup h_2$ in dual (relative) handle decomp.

Using uniqueness of 3-handle attachment [cf. Step 3 in pf of Thm 2 in Lecture 5] gluing $S^1 \times D^2$ is determined by att. sphere of the 2-h which is $\varphi(\{1\} \times \partial D^2) = p\mu_K + q\lambda_K$.

3) $K_{p/q}$ is a closed, or., well def'd 3-mfd. □

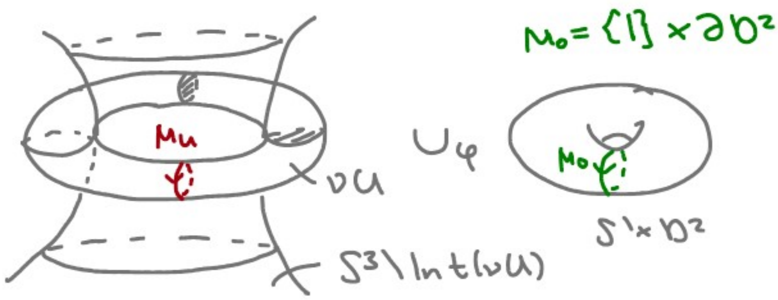
Ex: 1) $S^3_K(\frac{1}{0}) \cong_{C^\infty} S^3$ $\forall$ knots K

e.g.  $\infty = \frac{1}{0} \cong_{C^\infty} S^3$

why? $\frac{1}{0} \mathrel{\hat{=}} 1 \cdot \mu_K + 0 \cdot \lambda_K = \mu_K$

i.e. we map meridian $\{1\} \times \partial D^2$ of $S^1 \times D^2$ to meridian μ_K of K

e.g. $K = U$:



glue $S^3 \setminus \mathrm{Int}(\nu U) \cup S^1 \times D^2$

$\mu_U \leftarrow \underbrace{\{1\} \times \partial D^2}_{\mu_0}$

i.e. we glue back a solid torus the way we removed it!

2) $\bigcirc^{p/q} \cong_{C^\infty} L_{p,q}$ Corrected version:

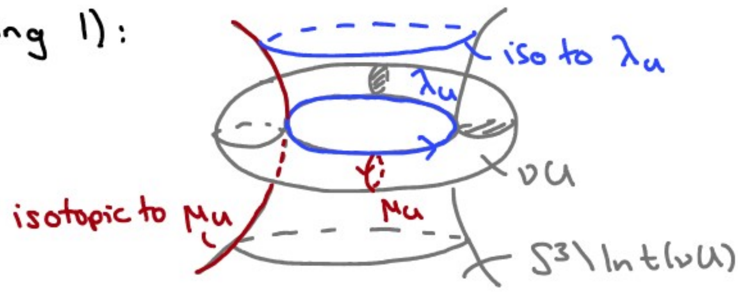
Why? $\bigcirc^{p/q} := S^3 \setminus \text{Int}(\nu U) \cup S^1 \times D^2$

$p\mu_U + q\lambda_U \longleftarrow \{1\} \times \partial D^2$

Two steps: 1) Remove $\text{Int}(\nu U)$ from S^3

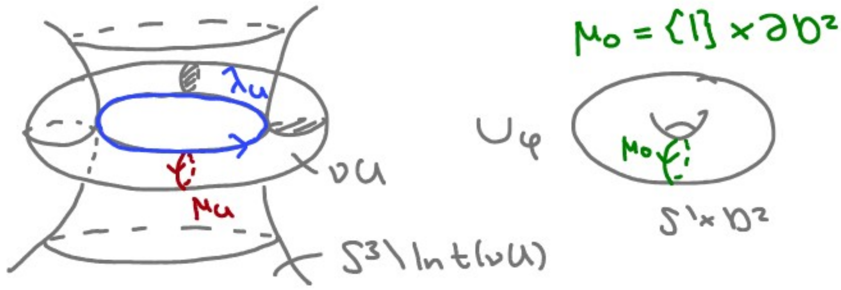
2) Glue back in $S^1 \times D^2$

Regarding 1):



Note that a basis for $\pi_1(T^2) \cong \pi_1(\partial(S^3 \setminus \text{Int}(\nu U)))$ is given by $(\mu_1, \lambda_1) = (\lambda_U, \mu_U)$.

Regarding 2):



We now glue back $S^1 \times D^2$ along

$$\varphi: \partial(S^1 \times D^2) \longrightarrow \partial(S^3 \setminus \text{Int}(\nu U))$$

$$\mu_0 = \{1\} \times \partial D^2 \longmapsto p\mu_U + q\lambda_U$$

$$= p\lambda_1 + q\mu_1 \leftrightarrow \underbrace{\begin{pmatrix} q & p \\ \cdots & \cdots \end{pmatrix}}_{\in GL_2(\mathbb{Z})} \begin{pmatrix} \mu_1 \\ \lambda_1 \end{pmatrix}$$

which corresponds to $\mu_2 \mapsto p\lambda_1 + q\mu_1 = \underbrace{\begin{pmatrix} q & p \\ \cdots & \cdots \end{pmatrix}}_{\in GL_2(\mathbb{Z}) \cong \pi_1(T^2)} \begin{pmatrix} \mu_1 \\ \lambda_1 \end{pmatrix}$

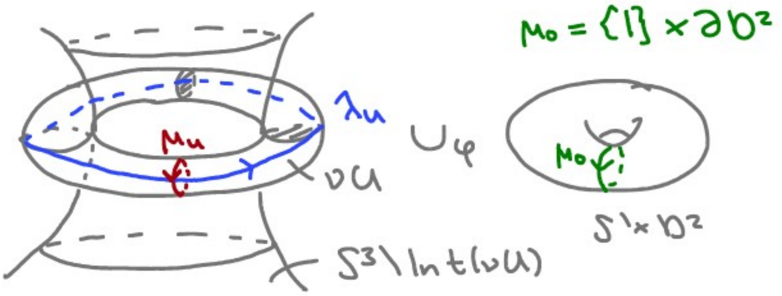
This gives $L_{p,q}$, see Lecture 7 [corrected version of notes].

3) $\bigcirc^{1/q} \cong_{C^\infty} L_{1,q} \cong S^3$ follows from 2)

4) $\bigcirc^{0=0/1} \cong_{C^\infty} L_{0,1} \cong S^1\times S^2$

<u>Why?</u> $S^3\setminus \mathrm{Int}(\nu U)\cup_\varphi S^1\times D^2$

$$\tfrac{0}{1} \,\hat{=}\, 0\cdot\mu_U + 1\cdot\lambda_U \overset{\varphi}{\hookleftarrow} \mu_0 := \{1\}\times\partial D^2 \subseteq S^1\times D^2$$
$(p=0, q=1)$



glue $S^3\setminus\mathrm{Int}(\nu U)\cup S^1\times D^2$

$$\lambda_U \hookleftarrow \underbrace{\{1\}\times\partial D^2}_{\mu_0}$$

i.e. we glue two solid tori along their meridians $\leadsto S^1\times S^2$

<u>Thm 1</u> (Lickorish '60, Wallace '62, Dehn)

Every closed, oriented (orientable enough), connected 3-manifold M^3 can be obtained as Dehn surgery on an ordered link L, i.e. $\exists$ link $L = L_1\cup\dots\cup L_m$ in S^3 s.t.

$$M^3 \cong_{C^\infty} S^3_L(r_1;\dots;r_m) := S^3\setminus \mathrm{Int}\,\nu L \cup_\varphi \bigsqcup_{i=1}^m S^1\times D^2_i,$$
$$p_i\mu_i + q_i\lambda_i \overset{\varphi}{\hookleftarrow} \{1\}\times\partial D^2_i$$

where $r_i \in \mathbb{Q}\cup\{\infty\}$ is the <u>surgery coefficient</u> to L_i, $i=1,\dots,m$
$r_i = \frac{p_i}{q_i}$ for coprime p_i, q_i

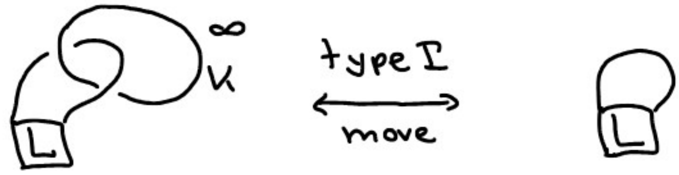
& $\nu L = \bigsqcup_{i=1}^m \nu L_i$ w/ meridians μ_i, Seif. long. λ_i

[small enoug s.t. disjoint]

In addition, one can find such a link L s.t. all surgery coefficients $r_i \in \mathbb{Z}$, $i=1,\dots,m$, & even s.t. $r_i = \pm 1$, $i=1,\dots,m$, & s.t. all components of L are unknotted.

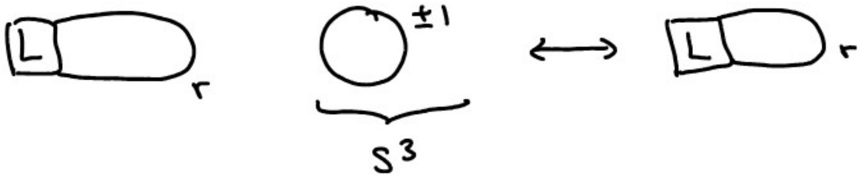
(We call Dehn surgery along L w/ all $r_i \in \mathbb{Z}$ <u>integral</u> surgery.)

<u>Ex:</u>

- 

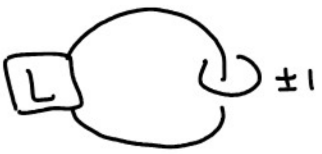
$\Rightarrow$ diffeomorphic mfds

  addition / deletion of a component w/ surg. coeff. $\underline{\underline{\infty}}$ potentially knotted / linked!

- 

$\Rightarrow$ diffeo mfds

  addition / deletion of <u>unknotted</u> comp. w/ surg. coeff. $\underline{\underline{\pm 1}}$ that is <u>unlinked</u> from the other components

- 

here can't remove the comp. w/ coeff. ± 1 in general (if not <u>split</u> link)

<u>Ex:</u> 

$\cong_{C^\infty} S^1 \times S^2$ (Rolfsen, 9.G.7)

but 

$\cong S^3$ (see above)

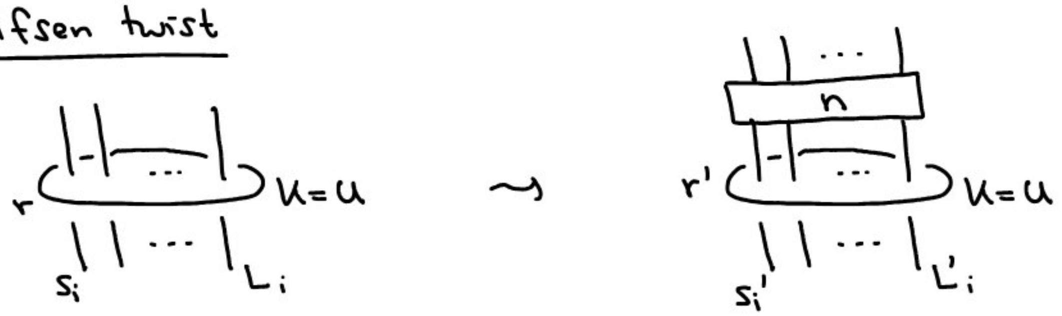
<u>Question during break:</u> Is one component ($m=1$) in Thm 1 always enough?

<u>Answer:</u> <u>No!</u> e.g. 

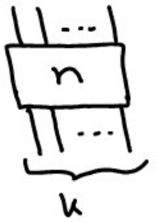
$\cong_{C^\infty} T^3$

and homology considerations show that 3 components are needed for T^3

Rolfsen twist



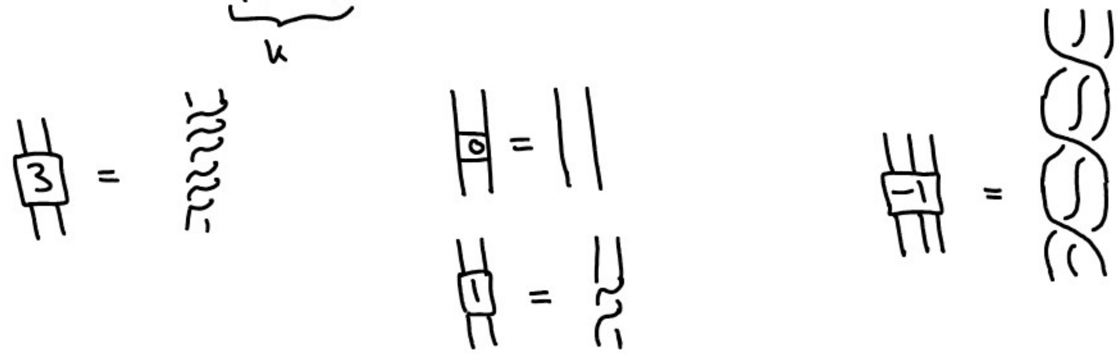
where



= n full twists on k strands

$n \in \mathbb{Z}$

e.g.



Change of surgery coefficients: $r' = \dfrac{1}{n + \frac{1}{r}}$

$s_i' = s_i + n \, \mathrm{lk}(L_i, K)^2$

References for Rolfsen twist:

- p. 162, Gompf - Stipsicz
- Prop. 9.H.2, p. 264-267, Rolfsen
- §14.8 & §16.5, Prasolov - Sossinsky

Thm 2 (Kirby, Fenn-Rourke, Rolfsen)

References:
- Thm 5.3.6 & Prop. 5.3.10, Gompf-Stipsicz
- Rem 9.I.7, Prop. 9.H.2, Rolfsen
- Thm 19.5, Prasolov-Sossinsky

Two Dehn surgery presentations $(L; r_1; \dots; r_m)$ & $(L'; s_1; \dots; s_n)$ describe diffeomorphic or., conn., cld 3-mfd if and only if they are related by a finite sequence of moves

(I) addition / deletion of  ∞

(link components w/ surg. coeff. ∞)

(II) Rolfsen twists