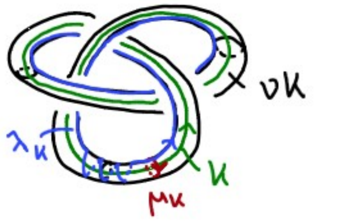
Lecture 9

Last time: Dehn surgery along a link $L = L_1 \cup \ldots \cup L_m$ w/ surgery coeff. $r_i = \frac{p_i}{q_i} \in \mathbb{Q} \cup \{\infty\}$ for L_i, $i = 1, \ldots, m$ is

p_i, q_i coprime



$lk(\mu_K, K) = +1$

$lk(\lambda_K, K) = 0$

$$S^3_L(r_1; \ldots; r_m) := S^3 \setminus \mathrm{Int}(\nu L) \cup_\varphi \coprod_{i=1}^{m} S^1 \times D^2_i$$

$$p_i \mu_i + q_i \lambda_i \xleftarrow{\varphi} \{1\} \times \partial D^2_i$$

where $\mu_i = \mu_{L_i}$ meridian of L_i

$\lambda_i = \lambda_{L_i}$ Seifert longitude of L_i

[Rem: can also think of first attaching 2-handles $D^2_i \times S^1$ along $p_i \mu_i + q_i \lambda_i$ & then fill the result w/ 3-balls]

Ex:  $\infty = \frac{1}{0}$ $\cong_{C^\infty} S^3$

$K =$ fig. 8

$\frac{1}{0} \mathrel{\hat{=}} 1 \cdot \mu_1 + 0 \cdot \lambda_1 = \mu_K$

i.e. $\mu_0 = \{1\} \times \partial D^2 \mapsto \mu_K$

(i.e. remove $\nu K \cong S^1 \times D^2$ and glue back $S^1 \times D^2$ same way)

Ex:  p/q $\cong_{C^\infty} L_{p,q}$

$K = U$ unknot

For details why, see the notes for Lecture 8.

Ex: Consequence of above ex:

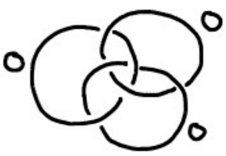
 $1/q$ $\cong L_{1,q} \cong S^3$

 $0/1$ $\cong S^1 \times S^2$

 $3/7$ $\cong L_{3,7} \cong L_{3,1}$

More ex: 1)  $\cong_{C^\infty} S^3$ (Exercise 5 on Sheet 4)

positive Hopf link (isotopic to ) (ex. from Lect 7)

2) (without proof)  $\cong_{C^\infty} T^3 = S^1 \times S^1 \times S^1$

[Rem: this ex can be used to show that one link component is not necessarily enough when representing a given 3-mfd as Dehn surgery along a link]

← last time

Thm 1 (Lickorish '60, Wallace '62, Dehn)

Every closed, oriented, connected 3-manifold M^3 can be obtained as Dehn surgery on an ordered link L, i.e. $\exists$ link $L = L_1 \cup \dots \cup L_m$ in S^3, $r_i \in \mathbb{Q} \cup \{\infty\}$ s.t. $M^3 \cong_{C^\infty} S^3_L(r_1; \dots; r_m)$.

In addition, one can find such a link L s.t. all surgery coefficients $r_i = \pm 1$, $i = 1, \dots, m$ & s.t. all L_i are unknotted.
& s.t. all components of L are unknotted.

isotopic to unknot
$L_i \underset{iso}{\cong} U$

For simplicity, we will only prove the existence of a homeomorphism

$$M^3 \cong_{C^0} S^3_L(r_1; \dots; r_m).$$

For the proof of Thm 1, we'll use

Thm 3 (Dehn 1938, Lickorish 1962 & 1964)

Let F be a closed, orientable, conn. surface of genus $g \geq 1$.
Then the mapping class group of F

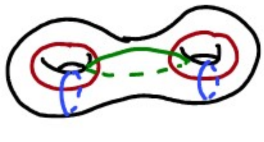
$$\mathrm{mcg}(F) := \mathrm{Homeo}^+(F)/\text{isotopy}, \quad \text{where}$$

$$\mathrm{Homeo}^+(F) := \{\varphi : F \to F \text{ or. pres. homeo}\}$$

is generated by Dehn twists along $3g-1$ curves.

$2g + g - 1 = 3g - 1$ curves

e.g.



$g = 2$, $3g - 1 = 5$

Consequence: Every $\varphi \in \mathrm{Homeo}^+(F)$ is isotopic to a product of Dehn twists.

[Note that if $k_1, \dots, k_r \in \mathrm{Homeo}(F)$ are isotopic to id_F & $T_1, \dots, T_r \in \mathrm{Homeo}(F)$ are arbitrary, then $k_1 \circ T_1 \circ \dots k_r \circ T_r$ is isotopic to $T_1 \circ \dots \circ T_r$ (pf: exer).]

Rem: There are many equivalent definitions of mcg (F), see e.g.

B. Farb and D. Margalit, A Primer on Mapping Class Groups , p. 44-45 :

$$\mathrm{mcg}(F) \cong \underset{\substack{\uparrow \\ \text{gp w/ composition as} \\ \text{multiplication}}}{\mathrm{Homeo}^+(F)} / N(F), \quad N(F) := \{\varphi \in \mathrm{Homeo}^+(F) \mid \varphi \underset{iso}{\sim} \mathrm{id}_F\} \underset{\substack{\uparrow \\ \text{normal subgp}}}{\triangleleft} \mathrm{Homeo}^+(F)$$

$$\cong \pi_0(\underset{\text{w/ compact-open topology}}{\mathrm{Homeo}^+(F)})$$

We won't prove Thm 3 in class. A pf (← long!) can be found e.g. in the book B. Farb and D. Margalit, A Primer on Mapping Class Groups [see Thms 4.1, 4.11, 4.13] which also contains much more material on mapping class groups of surfaces, Dehn twists, ...

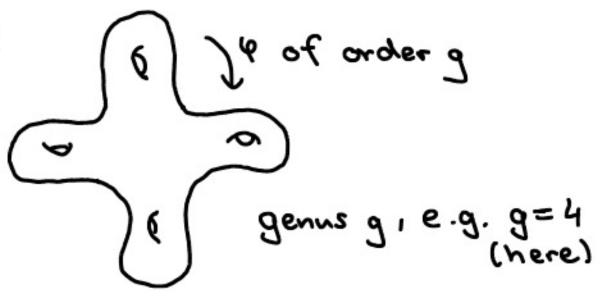
The original sources by Lickorish are

- W. B. R. Lickorish, "A representation of orientable combinatorial 3-manifolds." Ann. of Math. (2) 76 1962 531—540. (finite generation of mcg(F) & pf of Thm 1 in PL category)
- W. B. R. Lickorish, "A finite set of generators for the homotopy group of a 2-manifold", Proc. Cambridge Philos. Soc. 60 (1964), 769–778 (specific generating set for mcg(F)).

Dehn twists (along essential simple closed curves) have infinite order in mcg(F).

Other examples of elements of mcg (F):

Ex:



Other notation for mcg (F) in the literature: MCG(F)
Mod(F)
$\mathcal{M}(F)$
Map(F)
⋮

Def: (Dehn twist) Let c be a simple closed curve on F.

Choose an orientation on c & an or. pres. diffeo / homeo

$\nu c \cong S^1 \times [0,1]$ s.t. $c \mapsto S^1 \times \{\frac{1}{2}\}$.

tubular nbhd

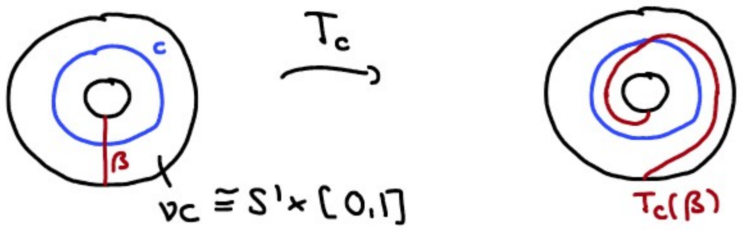
A (right-handed) Dehn twist along c, denoted $T_c \in \mathrm{Homeo}^+(F)$,

is defined by

$$T_c|_{F \setminus \mathrm{Int}(\nu c)} := \mathrm{id}_{F \setminus \mathrm{Int}(\nu c)}$$

$$T_c|_{\nu c \cong S^1 \times [0,1]} : \nu c \cong S^1 \times [0,1] \circlearrowleft$$

$$(e^{2\pi i \theta}, t) \mapsto (e^{2\pi i (\theta + t)}, t)$$

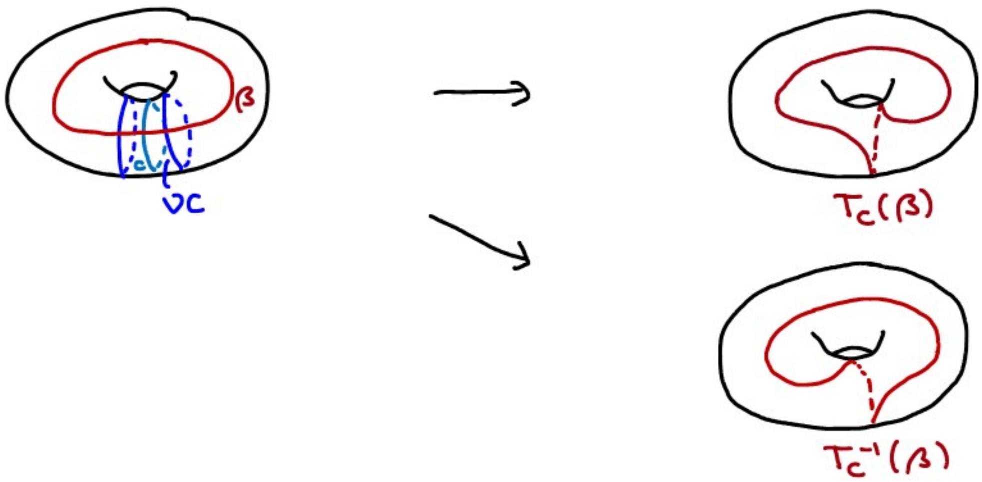


"walk on β on νc & turn right before meeting c"

T_c^{-1} is a left-handed Dehn twist.

Rem: Only depends on orientation of F, not on or. of c.

Ex:

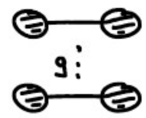


Pf of Thm 1: [See Rolfsen, 9.I.5 & 9.I.4]

Let M^3 be an or., cld, conn. 3-mfd. $\underset{(\S 3.1)}{\overset{\text{Thm 1, Lect 4}}{\Longrightarrow}}$ $\exists$ Heegaard splitting of M, $M = H_1' \cup_{\varphi'} H_2'$, $\varphi' : \partial H_2' \underset{C^\infty}{\overset{\cong}{\longrightarrow}} \partial H_1'$ or. rev. diffeo, of genus $g = g(\partial H_1') = g(\partial H_2')$.

Let $S^3 = H_1 \cup_\varphi H_2$ be a Heegaard splitting of S^3 of genus g.

$\varphi : \partial H_2 \to \partial H_1$
$\cong_{C^\infty}$
or. rev.

(e.g. the Heeg. splitting w/ Heeg. diagram )

Prop 1, Lect 4 $\Rightarrow$ $\exists$ a homeo $h : H_1 \to H_1'$.

(Any handle body is diffeo to $\natural_g (S^1 \times D^2)$.)

$\partial H_2 \xrightarrow{\varphi} \partial H_1 \xrightarrow{h|_{\partial H_1}} \partial H_1' \xrightarrow{(\varphi')^{-1}} \partial H_2'$

Key Claim: The homeo $f := (\varphi')^{-1} h \varphi : \partial H_2 \to \partial H_2'$ extends to a homeo $\bar{f}$ of $H_2 \setminus (\mathring{V}_1 \cup \ldots \cup \mathring{V}_r)$ onto $H_2' \setminus (\mathring{V}_1' \cup \ldots \cup \mathring{V}_r')$ where $V_1, \ldots, V_r \subset H_2$ & $V_1', \ldots, V_r' \subset H_1$ are solid tori & $\mathring{V} := \mathrm{Int}(V)$.

Key Claim $\Rightarrow$ The homeo $h : H_1 \to H_1'$ extends to a homeo

$$\bar{h} : S^3 \setminus (\mathring{V}_1 \cup \ldots \cup \mathring{V}_r) \longrightarrow M^3 \setminus (\mathring{V}_1' \cup \ldots \cup \mathring{V}_r').$$

Γ
$$\begin{array}{ccccccc}
S^3 \setminus (\mathring{V}_1 \cup \ldots \cup \mathring{V}_r) & = & (H_1 \cup_\varphi H_2) \setminus (\mathring{V}_1 \cup \ldots \cup \mathring{V}_r) & = & H_1 & \cup & (H_2 \setminus (\mathring{V}_1 \cup \ldots \cup \mathring{V}_r)) \\
\bar{h} \downarrow \cong_{C^0} & & & & h \downarrow \cong_{C^0} & & \bar{f} \downarrow \cong_{C^0} \\
M^3 \setminus (\mathring{V}_1' \cup \ldots \cup \mathring{V}_r') & = & (H_1' \cup_{\varphi'} H_2') \setminus (\mathring{V}_1' \cup \ldots \cup \mathring{V}_r') & = & H_1' & \cup & (H_2' \setminus (\mathring{V}_1' \cup \ldots \cup \mathring{V}_r'))
\end{array}$$
$\lrcorner$

$\Rightarrow M^3 \cong_{C^0} (S^3 \setminus (\mathring{V}_1 \cup \ldots \cup \mathring{V}_r)) \cup (\mathring{V}_1' \cup \ldots \cup \mathring{V}_r') \Rightarrow 1^{\text{st}}$ part of Thm 1.

Pf of Key Claim shows that $\bar{h} : \partial V_i \to \partial V_i'$ & preimage of a meridian of V_i' is meridian $\pm$ longitude of V_i.

$\Rightarrow M^3$ is ± 1 surgery on solid tori $V_1, \ldots, V_r$ (i.e. along the link given by the cores of the solid tori V_i)

Pf of Key Claim:

NTS: $\exists$ solid tori $V_i \subset H_2$, $V_i' \subset H_2'$, $i=1,\dots,r$ s.t. the homeo $f: \partial H_2 \to \partial H_2'$ extends to a homeo
$$\bar{f}: H_2 \setminus (\mathring{V}_1 \cup \dots \cup \mathring{V}_r) \longrightarrow H_2' \setminus (\mathring{V}_1' \cup \dots \cup \mathring{V}_r').$$

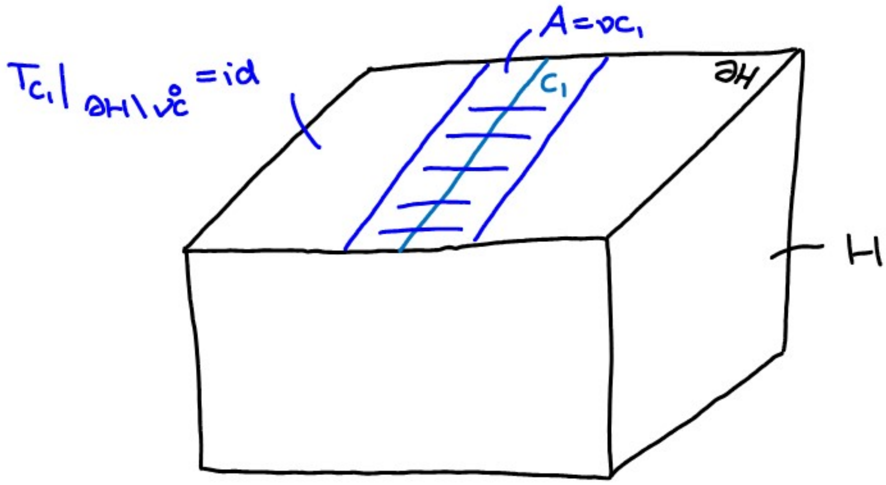
We have $H_2 \cong_{C^0} H_2'$. WLOG we can suppose $H := H_2 = H_2'$, f or. preserving.

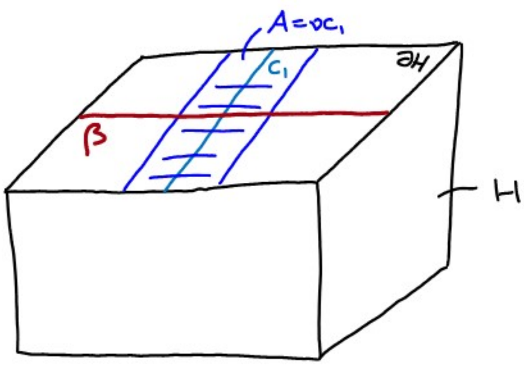
If f is isotopic to id, it's easy to find an extension on (all of!) H. $(*)$
["move only a collar of ∂H"]

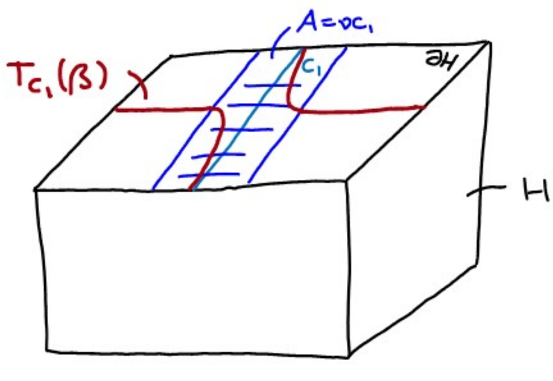
Consequence of Thm 3 $\Longrightarrow$ Can assume $f = T_r \circ \dots \circ T_1$ for Dehn twists $T_i = T_{c_i} \in \mathrm{Homeo}^+(\partial H)$.
[Up to isotopy; use previous step $(*)$.]

Slogan: "perform Dehn twist along c_1 at expense of cutting out solid torus $V_1 \subset H = H_2$".

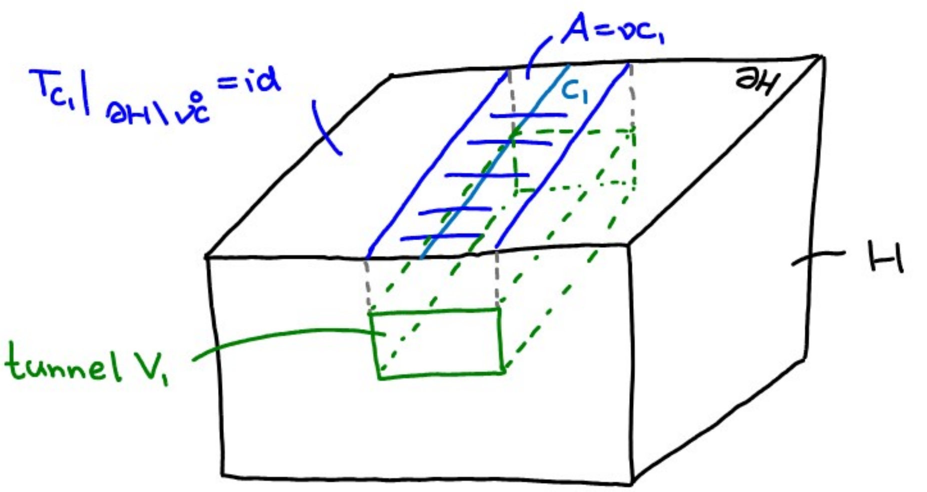
Consider T_{c_1}. It's the identity off an annular nbhd $A = \nu c_1 \cong S^1 \times [0,1]$ of c_1 in ∂H.



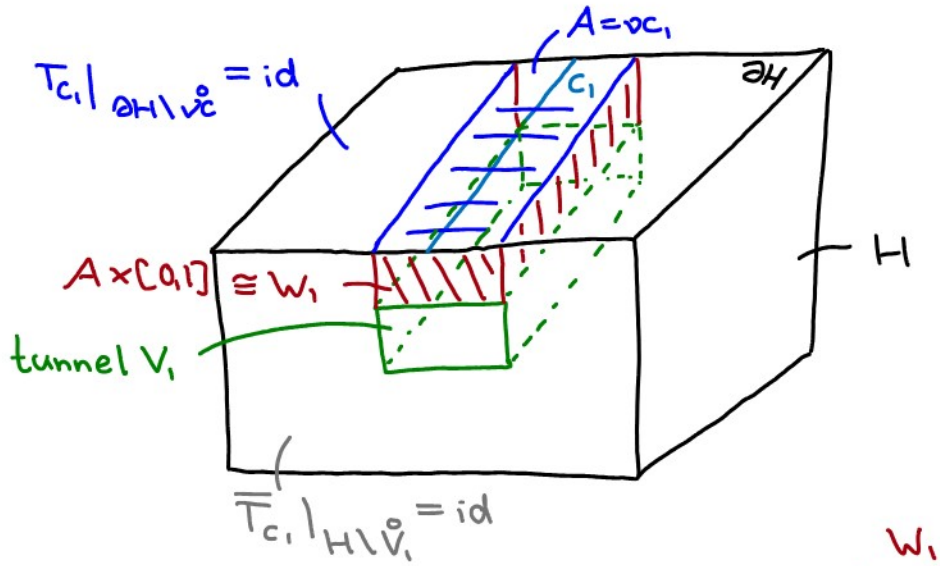




Imagine a <u>tunnel</u> excavated from H just under A.
↳ solid torus V_1 w/in a collar of ∂H



The region between V_1 and A is a copy of $A \times [0,1]$, call it W_1.



W_1 area that gets twisted

Extend $T_{c_1} \in \mathrm{Homeo}^+(\partial H)$ by
- $\boxed{T_{c_1} \times id \text{ on } W_1}$
- id elsewhere on $H \setminus \mathring{V}_1$

$$\rightsquigarrow \overline{T}_{c_1} : H \setminus \mathring{V}_1 \to H \setminus \mathring{V}_1$$

Similarly for $c_2, \dots, c_r$:

Extend T_{c_2} to $\overline{T}_{c_2} : H \setminus \mathring{V}_2 \to H \setminus \mathring{V}_2$ by excavating slightly deeper than before so that V_2 misses V_1 & $\overline{T}_{c_1}$ is the identity on V_2.

Inductively, we find solid tori $V_1, \dots, V_r$ & extensions

$\overline{T_{c_i}} : H \setminus \mathring{V}_i \to H \setminus \mathring{V}_i$ s.t. $\overline{T_{c_i}}$ is fixed on V_j for $i<j$. Then

$$\overline{f} := \overline{T_{c_r}} \circ \dots \circ \overline{T_{c_2}} \circ \overline{T_{c_1}} : H \setminus (\mathring{V}_1 \cup \dots \cup \mathring{V}_r) \to H \setminus (\mathring{V}_1' \cup \dots \cup \mathring{V}_r')$$

where $V_r' := V_r$,
$V_i' := T_{c_r} \circ \dots \circ T_{c_{i+1}} (V_i)$ for $i<r$,

is the required extension of f. $\square$