The alternation number and the Upsilon-invariant at 1 of positive 3-braid knots

K-OS, February 17

Thm (T. '21):

Let K be a knot in S^3 that is the closure of a positive 3-braid.

Then $\mathrm{alt}(K) = \tau(K) + \upsilon(K) = g(K) + \upsilon(K) = \underbrace{\mathrm{dalt}(K) = g_T(K) = A_s(K)}_{\text{other alternating distances}}$.

alternation number

$\upsilon(K) = \Upsilon_K(1)$ given by explicit formulas

I) Positive 3-braid knots

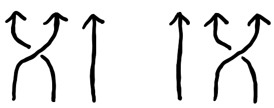
II) Alternation number

III) Thm & sketch of proof

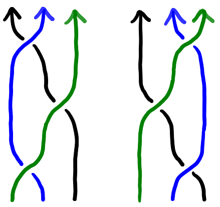
I) Positive 3-braid knots

$B_3 = \langle a, b \mid aba = bab \rangle$ braid group on 3-strands

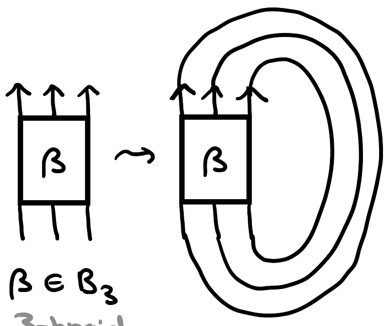
braid relation:



a b



$aba = bab$



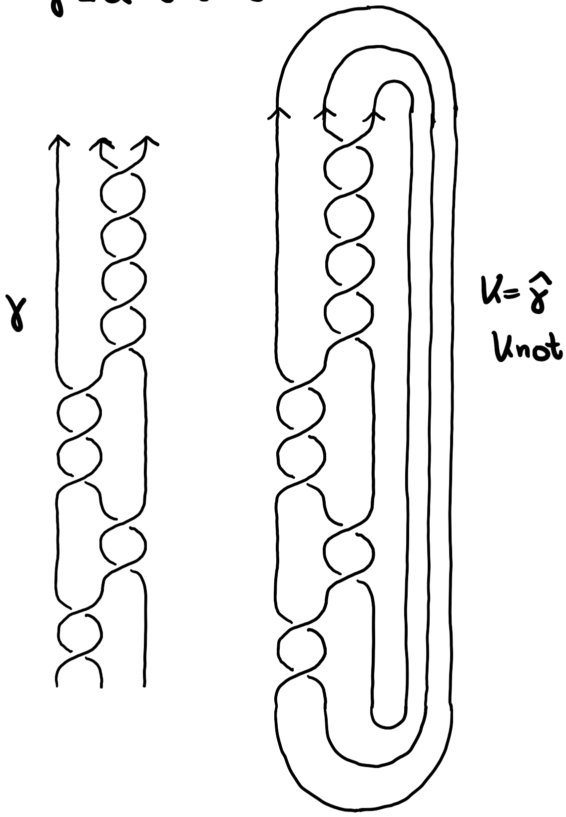
$\beta \in B_3$
3-braid

closure $\hat{\beta}$ ⇝ knot or link

A **positive 3-braid** is $\beta \in B_3$ that can be written as a word in only positive generators a, b, e.g. $\gamma = a^2 b^2 a^3 b^5$ (no a^{-1}, b^{-1})

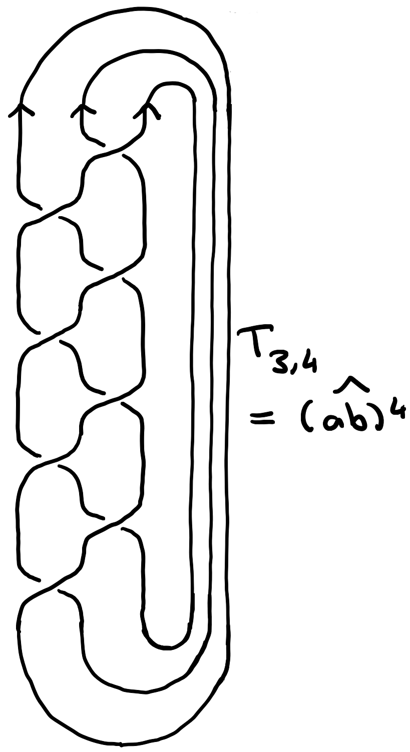
A **(positive) 3-braid knot** is a knot that can be represented as the closure of a (positive) 3-braid.

Ex: • $\gamma = a^2 b^2 a^3 b^5$



• torus knots $T_{3,3n+k}$, $n \geq 0$, $k \in \{1,2\}$

e.g.



Fact: (slice-Bennequin inequality)

$$g(K) = \frac{wr(\beta) - n + 1}{2} \quad \text{if } K = \hat{\beta} \text{ is a knot for a } \underline{\text{positive}} \text{ braid } \beta \in B_n$$

Bennequin '83

strongly quasipositive enough

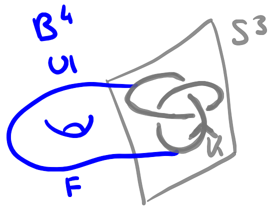
$$= g_4(K) = \tau(K)$$

Rudolph '93, Kronheimer–Mrowka '93

Livingston '04, Plamenevskaya '04, Hedden '10

where $g_4(K) = \min\{\text{genus}(F) \mid F$ or., conn., cpct smooth surface (smoothly embedded) in S^3 (B^4) w/ or. boundary K in $S^3 = \partial B^4\}$,

smooth (4-)genus of K

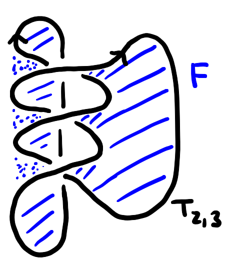


$\tau(K) \in \mathbb{Z}$ knot (concordance) invariant
(Ozsváth–Szabó '03, Rasmussen '03)

and $wr(\beta)$ = exponent sum for β, e.g. $wr(\underbrace{a^2 b^2 a^3 b^5}_{=\gamma}) = 12$

writhe

Ex:

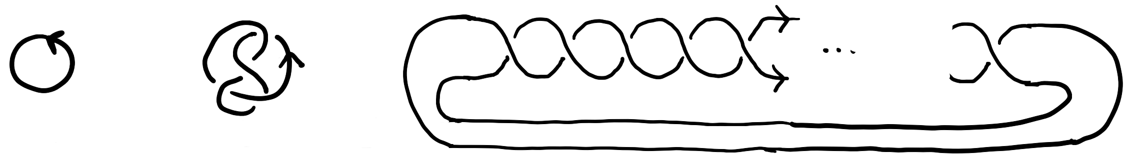


Ex: $g(\hat{\gamma}) = g_4(\hat{\gamma}) = \tau(\hat{\gamma}) = \frac{12-2}{2} = 5$

II) Alternation number

Recall: An alternating knot is a knot which has a diagram in which the crossings alternate between over- and underpasses as one travels along the knot.

Ex:



$T_{2,2n+1}$

aside: there are geometric descriptions in terms of spanning surfaces by Greene '17 & Howie '17.

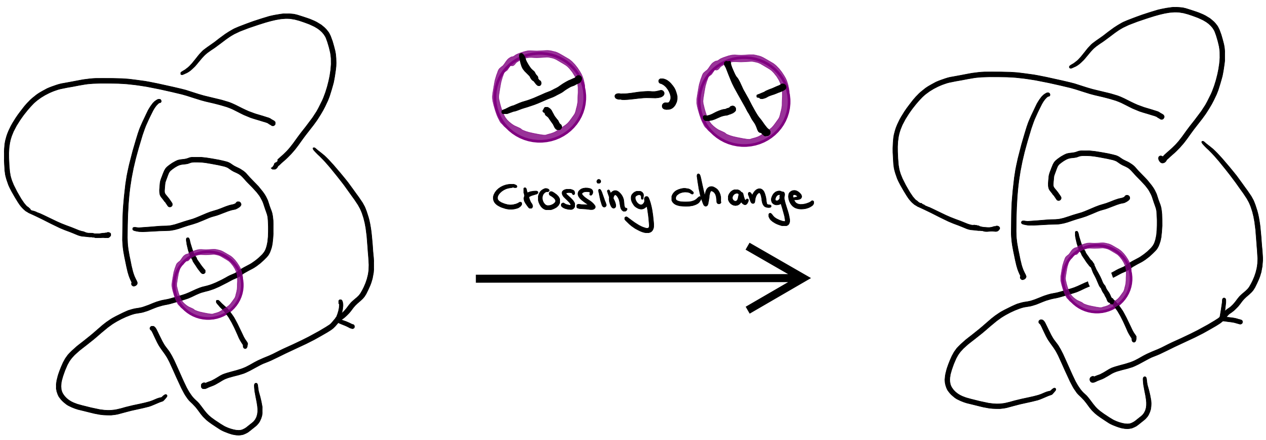
Def: alternation number (Kawauchi '10)

$$\mathrm{alt}(K) = d_{\mathrm{Gordian}}(K, \{\text{alternating knots}\})$$

$$= \min_{J \text{ alt. knot}} d_{\mathrm{Gordian}}(K, J)$$

= minimal number of crossing changes  needed to transform a diagram of K into the diagram of an alternating knot]

Ex:



$8_{19} = T_{3,4}$ alternating

$$\Rightarrow \mathrm{alt}(T_{3,4}) \leq 1$$

Lower bound on $\mathrm{alt}(K)$?

Fact (Livingston '04, Abe '09):

Signature (Trotter '62) Rasmussen '10

Let $\varphi_1, \varphi_2 \in \left\{\tau, \underbrace{-\upsilon, -\frac{\Upsilon(t)}{t}}, -\frac{\sigma}{2}, \frac{s}{2}\right\}$.

Ozsváth - Stipsicz - Szabó '17:

$\Upsilon_K: [0,1] \to \mathbb{R}$ continuous, piecewise linear function for any knot K

$\upsilon(K) := \Upsilon_K(1)$ upsilon of K (Upsilon at $t=1 \in [0,1]$)

Then $|\varphi_1(K) - \varphi_2(K)| \leq \text{alt}(K)$ for any knot K.

Properties of φ_1, φ_2: (as concordance homomorphisms)

- $\varphi_i(K \# J) = \varphi_i(K) + \varphi_i(J)$ $\forall$ knots K, J
- $|\varphi_i(K)| \leq g_4(K)$ $\forall$ knots K
- $\varphi_i(-K) = -\varphi_i(K)$ $i=1,2$

$-K$: mirror of reverse of K

__Key input for Fact__ (Ozsváth-Szabó, Rasmussen, Ozsváth-Stipsicz-Szabó):

For all alternating knots K, we have

$$\tau(K) = \frac{s(K)}{2} = -\upsilon(K) = -\frac{\Upsilon_K(t)}{t} = -\frac{\sigma(K)}{2} \qquad \text{for } t \in (0,1].$$

e.g. for $K = T_{2,3}$, $\tau(K) = \frac{s(K)}{2} = -\upsilon(K) = -\frac{\Upsilon_K(t)}{t} = -\frac{\sigma(K)}{2} = +1.$

Moreover, $0 \leq \varphi_i(K_+) - \varphi_i(K_-) \leq 1$, $i = 1,2$,

whenever K_+ & K_- are two knots that differ by changing a positive crossing in K_+ to a negative crossing in K_-.

__Ex:__ $\tau(T_{3,3n+1}) \overset{\text{by slice-Bennequin inequality}}{=} g_{(4)}(T_{3,3n+1}) = 3n$, $n \geq 0$

$\upsilon(T_{3,3n+1}) = -2n$ (Alexander polynomial determines $\Upsilon_K(t)$ for torus knots K)

$\Rightarrow |(\tau+\upsilon)(T_{3,3n+1})| = n \leq \text{alt}(T_{3,3n+1})$

e.g. $1 \leq \text{alt}(T_{3,4}) \leq 1 \Rightarrow \text{alt}(T_{3,4}) = 1$

III) Thm and sketch of proof

Thm (T. '21):

Let K be a knot in S^3 that is the closure of a positive 3-braid.

Then $\mathrm{alt}(K) = \tau(K) + \upsilon(K) = g(K) + \upsilon(K) = \mathrm{dalt}(K) = g_T(K) = A_s(K)$.

dealternating number

Turaev genus

min'l nr of double pt. singularities in generically immersed concordance from K to an alternating knot (Friedl–Livingston–Zentner '17)

Key input:

Thm 2 (T. '21): We determine $\upsilon(K)$ for all 3-braid knots K.

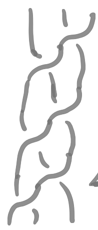
Remarks:
- For any 3-braid knot K, we determine $\mathrm{alt}(K)$ up to an error of 1.
- In some cases, $\mathrm{alt}(K)$ was determined by Abe–Kishimoto '10, Feller-Pohlmann–Zentner '18.

Prop: (Garside normal form) [$\leftrightarrow$ Murasugi: $\Delta^n a^{-p_1} b^{q_1} a^{-p_2} b^{q_2} \dots$ $p_i, q_i \geq 0$]

Let β be a $\boxed{\text{(positive)}}$ 3-braid. Then β is conjugate to one and only one of the 3-braids

$\hat{\beta}$ knots:

$\times$	(A)	$\Delta^{2e} a^p$, $e \in \mathbb{Z}, p \geq 0$	
torus knots $\leftarrow$ for p odd	(B)	$\Delta^{2e} a^p b$, $e \in \mathbb{Z}$, $p \in \{1,2,3\}$	
$\exists i,j$ s.t. p_i, q_j odd	(C)	$\Delta^{2e} a^{p_1} b^{q_1} a^{p_2} b^{q_2} \dots a^{p_r} b^{q_r}$	$e \in \mathbb{Z}$, $r \geq 1$, $p_i, q_i \geq 2$
$\exists i$: p_i or q_i odd	(D)	$\Delta^{2e+1} a^{p_1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} a^{p_r}$	$\boxed{e \geq 0}$



Δ^2

Here, $\Delta = aba = bab$; $\Delta^2 = (ab)^3$ (full twist) generates the center of B_3.

Sketch of pf of thm:

Ex: $K = \hat{\gamma}$ for $\gamma = a^2 b^2 a^3 b^5$ (case (C) of Prop. for $e=0, r=2$)

Outline:

Step I: Upper bound on $\mathrm{alt}(K)$, e.g. $\mathrm{alt}(\hat{\gamma}) \leq 1$

Step II: Lower bound on $\mathrm{alt}(K)$, e.g. $5 + \upsilon(\hat{\gamma}) \leq \mathrm{alt}(\hat{\gamma})$

Step III: Determine $\upsilon(K)$ for all (positive) 3-braid knots:

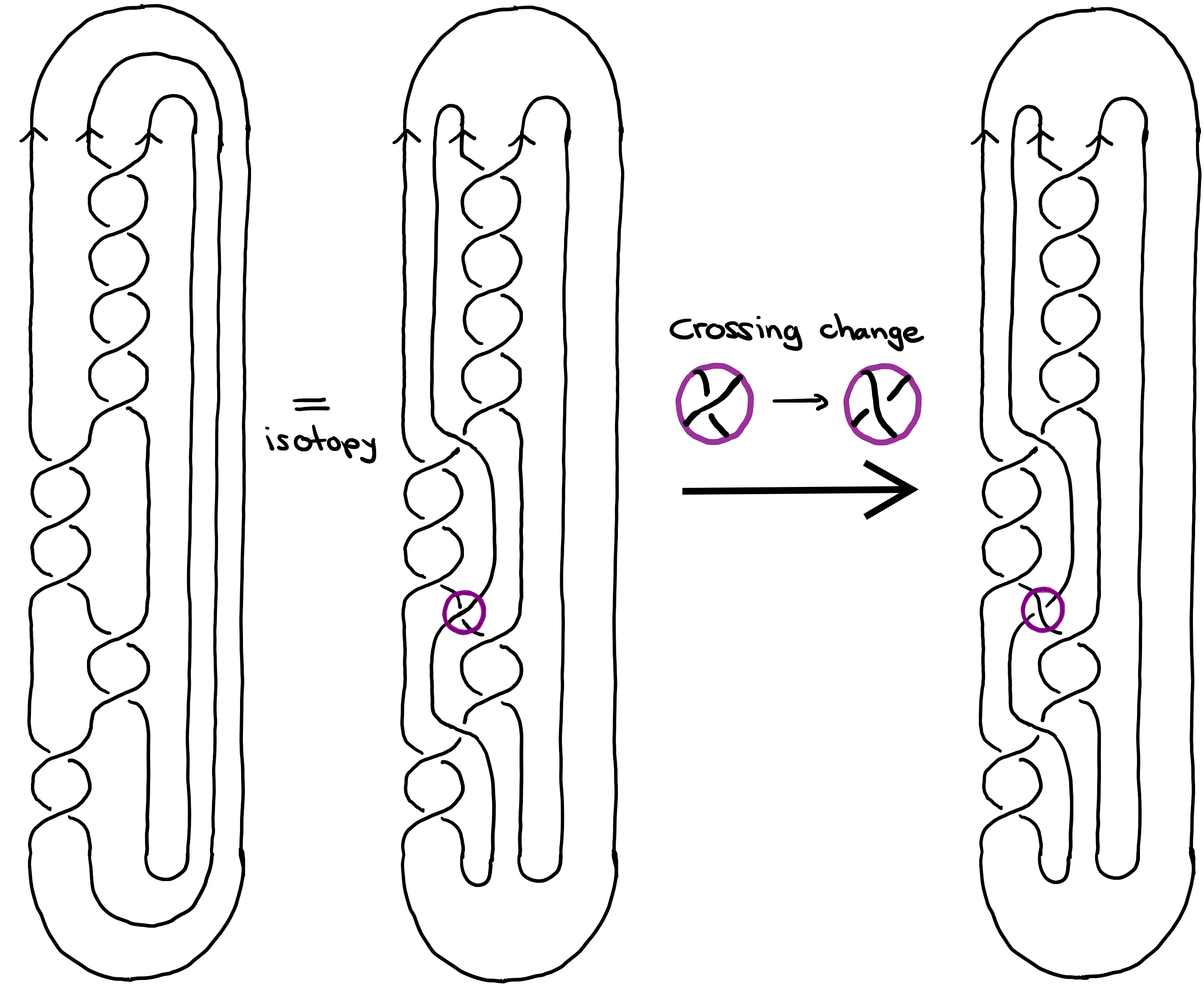
(C): $\upsilon(\hat{\beta}) = -\dfrac{\sum_{i=1}^{r}(p_i+q_i)}{2} + r - 2e$

$\Rightarrow \mathrm{alt}(\hat{\beta}) = r + e - 1$

(D): $\upsilon(\hat{\beta}) = -\dfrac{\sum_{i=1}^{r-1}(p_i+q_i) + p_r}{2} + r - 2e - \dfrac{3}{2}$

Step I: **Cl:** $\mathrm{alt}(\hat{\gamma}) \leq 1.$ $\gamma = a^2 b^2 a^3 b^5$

Pf:



alternating

$\Longrightarrow \mathrm{alt}(\hat{\gamma}) \leq 1$

Lemma (Abe-Kishimoto '10): (d) $\mathrm{alt}(\widehat{a^{p_1} b^{q_1} a^{p_2} b^{q_2} \ldots a^{p_r} b^{q_r}}) \leq r-1$

$p_i, q_i \geq 1,\ i = 1, \ldots, r.$

Step II: Lower bound from fact: $|\underbrace{\tau(K)}_{\substack{\shortparallel \\ g(K) = \frac{\mathrm{wr}(\gamma)-2}{2} = \frac{12-2}{2} = 5}} + \upsilon(K)| \leq \mathrm{alt}(K)$ $\quad K = \hat{\gamma}$, $\gamma = a^2 b^2 a^3 b^5$

$$\Longrightarrow \quad 5 + \upsilon(K) \leq \mathrm{alt}(K) \leq 1$$

Step III: Determine $\upsilon(K)$.

Idea: Find cobordism C between K and a knot T for which $\upsilon(T)$ is known, e.g. T = torus knot (L-space knot).

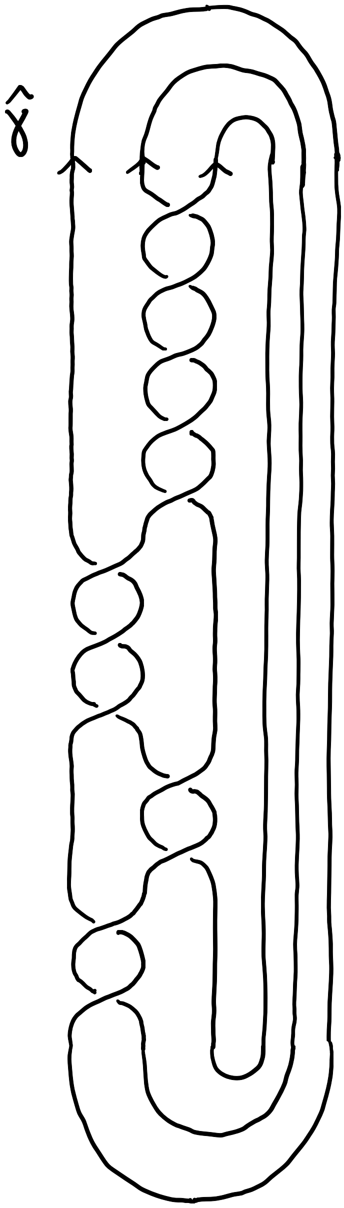
Then $|\upsilon(K) - \upsilon(T)| = |\upsilon(K \# -T)| \leq \underbrace{g_4(K \# -T)}_{= d_{cob}(K,T)} \leq g(C),$

$\uparrow$ genus of the cobordism C

so $\upsilon(K) \geq \upsilon(T) - g(C).$

In our ex: Cl: $v(\hat{\gamma}) \geq -4$. (Then $\mathrm{alt}(\hat{\gamma}) = 1 = g(\hat{\gamma}) + v(\hat{\gamma})$.)

$\gamma = a^2b^2a^3b^5 \xrightarrow[\text{in } \gamma \text{ to obtain } \alpha]{\text{delete 4 generators}} \alpha = a^2b^2ab^3$

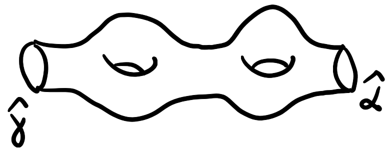


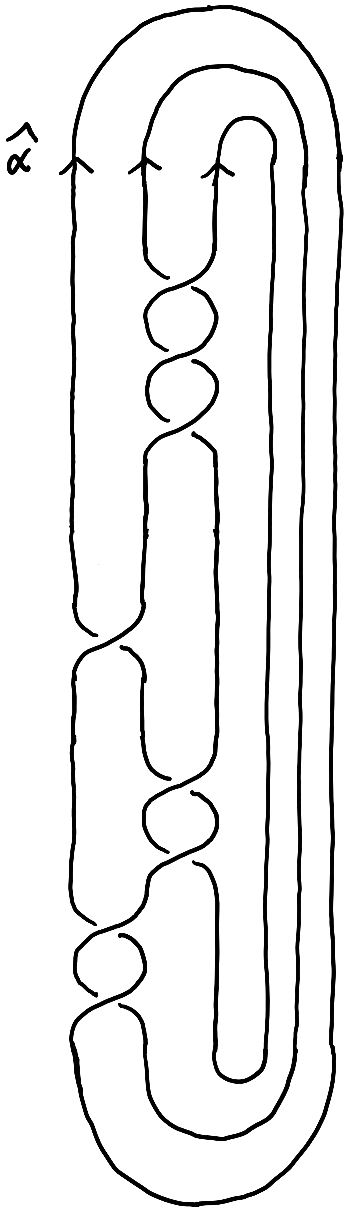
⊗ ≙ 4 saddle moves

$\longrightarrow$

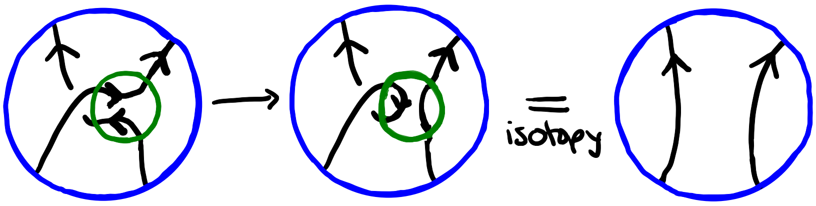
≙ cobordism of genus 2 between $\hat{\gamma}$ and $\hat{\alpha}$

(Euler characteristic argument)

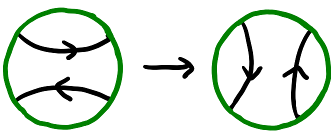


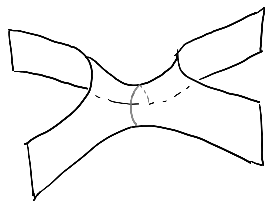


⊗ deleting a generator



using a saddle move





$\Rightarrow$ There is a cobordism of genus 2 between $\hat{\gamma}$ and $\hat{\alpha}$ $(\gamma = a^2b^2a^3b^5)$

We have $\alpha = a^2b^2ab^3 = (ab)^4$, $(a^2b^2ab^3 = a^2b\,\underline{bab}\,b^2 = a\underline{aba}\underline{bab}b = abababab$
$= (ab)^4$ using the braid relation $aba = bab)$

so $\hat{\alpha} = T_{3,4}$.

Hence $v(\hat{\gamma}) \geq \underbrace{v(T_{3,4})}_{=-2} - 2 = -2-2 = -4$ $\quad (v(T_{3,3n+1}) = -2n)$ $\quad \Rightarrow 5+4 \leq \mathrm{alt}(\hat{\gamma}) \leq 1 \Rightarrow \mathrm{alt}(\hat{\gamma}) = 1.$

□

Thm 2: Let β be a 3-braid s.t. $K=\hat{\beta}$ is a knot. Then

$$\upsilon(\hat{\beta}) = -\frac{\sum_{i=1}^{r}(p_i+q_i)}{2} + r - 2e \quad \text{if} \quad \beta \sim \Delta^{2e} a^{p_1} b^{q_1} a^{p_2} b^{q_2} \dots a^{p_r} b^{q_r} \quad (C)$$

$e \in \mathbb{Z}$, $r \geq 1$, $p_i, q_i \geq 2$

$$\upsilon(\hat{\beta}) = -\frac{\sum_{i=1}^{r-1}(p_i+q_i)+p_r}{2} + r - 2e - \frac{3}{2} \quad \text{if} \quad \beta \sim \Delta^{2e+1} a^{p_1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} a^{p_r} \quad (D)$$

$\upsilon(T_{3,3n+1}) = \upsilon(T_{3,3n+2}) + 1 = -2n$, $n \geq 0$

Rem: If K is a 3-braid knot which is not a torus knot, then $\upsilon(K) = \frac{\sigma(K)}{2}$. ← Erle '99

Cor. of Thm 2: Let K be a positive 3-braid knot. Then

$$r = g(K) + \upsilon(K) + 1 \quad =: r(K)$$

is minimal among all integers $r \geq 1$ s.t. K is the closure of a positive 3-braid $a^{p_1} b^{q_1} a^{p_2} b^{q_2} \dots a^{p_r} b^{q_r}$ for $p_i, q_i \geq 1$, $i \in \{1, \dots, r\}$.

If K, J are concordant positive 3-braid knots, then $r(K) = r(J)$.

Goal for the future: Understand concordance classes of positive 3-braid knots.

Thank you for your attention!