ETH zürich

D MATH

On notions of braid positivity and knot concordance

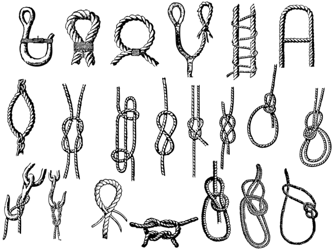
Paula Truöl

June 5, 2023

Outline

1. Introduction
   - Introduction to knots
   - Slice knots and knot concordance
   - Knots as closures of braids
2. Strongly quasipositive knots are concordant to infinitely many such knots
   - Strongly quasipositive knots and a conjecture by Baker
   - Theorem A: Statement and more context
   - Sketch of proof
3. On the concordance of positive 3-braid knots

Examples of knots



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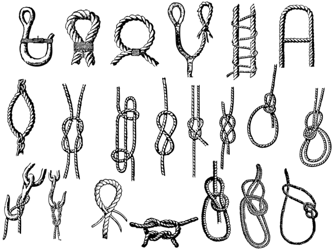

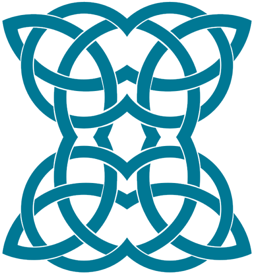

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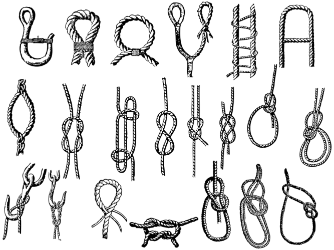

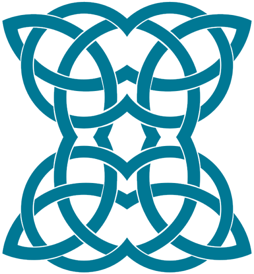

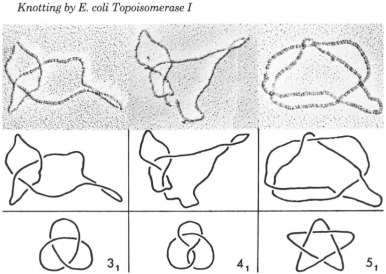

Dean–Stasiak–Koller–Cozzarelli 1985

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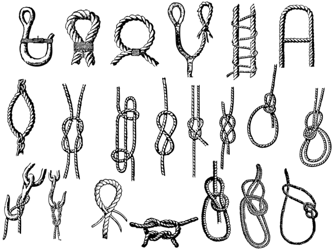

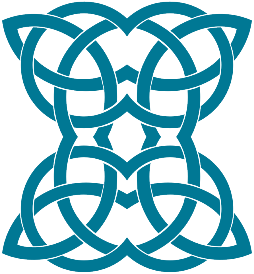

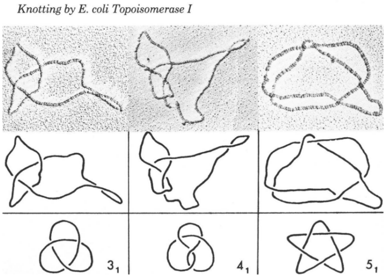

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Shakespeare, ∼1600

O time, thou must untangle this, not I.
It is too hard a knot for me t'untie.

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A **knot** is a **simple** closed curve in space, i. e. a curve with **no self-intersections**.

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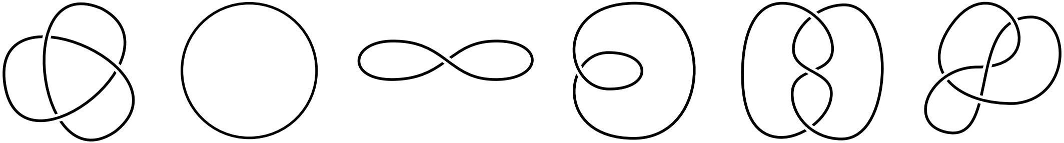

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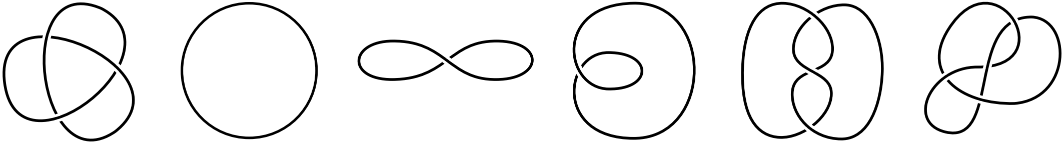

The second one is the **unknot**.

Low-dimensional topology and knot theory

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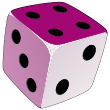

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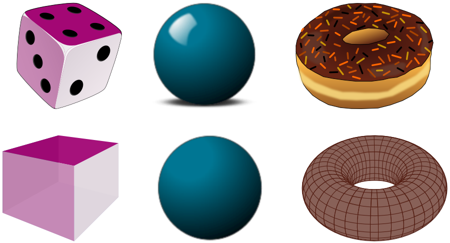

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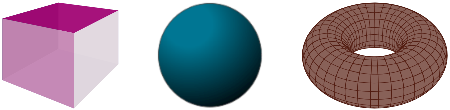

We can continuously deform a cube into a ball. But we cannot continuously deform a ball into a torus.

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Topology studies properties of spaces that are preserved under continuous deformations.
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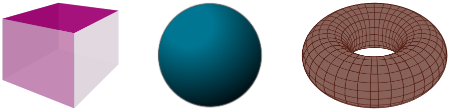

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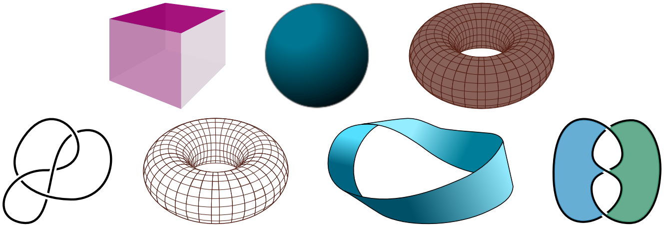

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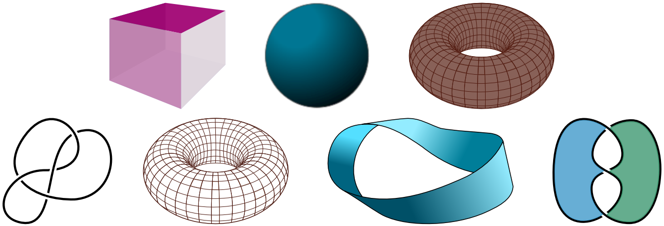

Knot theory studies knots in 3-space and their properties preserved under **continuous deformations**.

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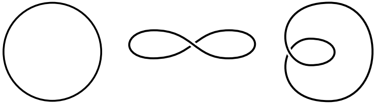
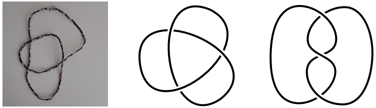
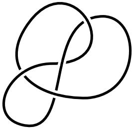


(a) The unknot.



(b) The trefoil knot.



(c) The figure-eight knot.

Figure: Examples of **(isotopy classes of)** knots.

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2) Knots can be used to reveal **exotic** structures of 4-manifolds.

Theorem (Moise, Stallings, Taubes, Gompf, Freedman 1960s–1980s)

*For $n \neq 4$, there is a unique smooth structure on $\mathbb{R}^n$. In contrast, there are **uncountably many smooth structures** on $\mathbb{R}^4$. This can be shown using the existence of topologically, but not smoothly **slice** knots.*

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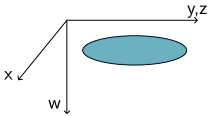

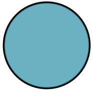

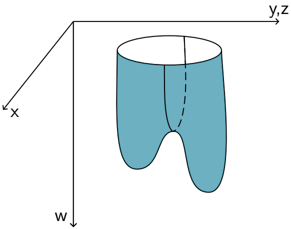

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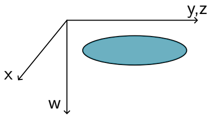

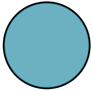

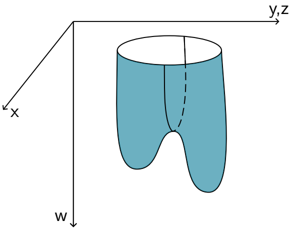

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A knot is isotopic to the unknot if and only if it bounds a disk in S^3.

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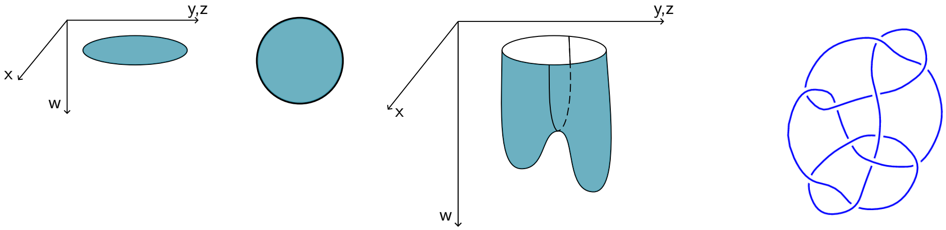

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Theorem (Piccirillo 2020)

The Conway knot is not slice.

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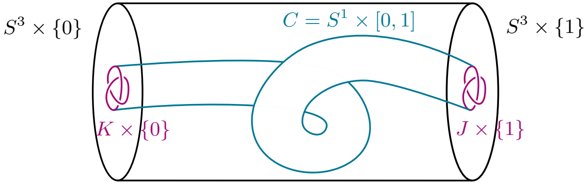
Motto: Concordance generalizes isotopy between knots to dimension 4.

Definition

Two knots K and J in S^3 are **concordant** if they cobound a smoothly embedded **cylinder** in $S^3 \times [0,1]$.



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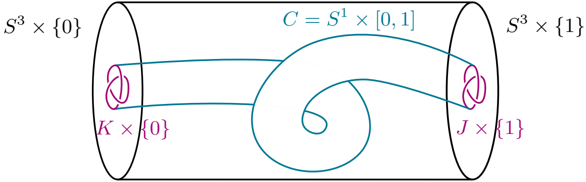

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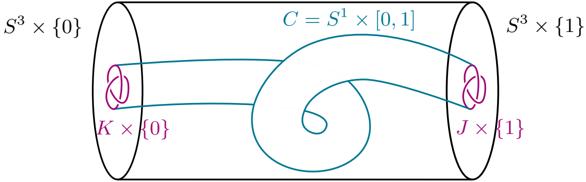

Concordance is an **equivalence relation**. The **concordance group** is the (countable abelian) group

$$\mathcal{C} = (\{\text{concordance classes of knots}\}, \text{connected sum}) \qquad \text{(Fox–Milnor 1966).}$$

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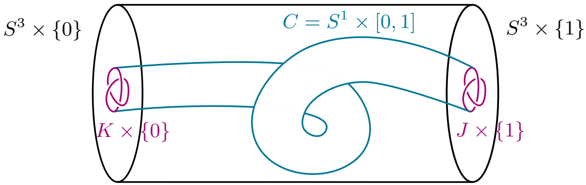

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Slice knots are concordant to the unknot and represent the identity element.

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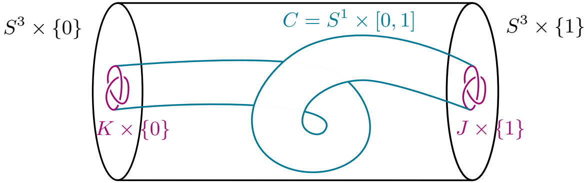

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Let K and J be knots in S^3. Then:

- K and J are isotopic knots. $\Rightarrow$ K and J are concordant.

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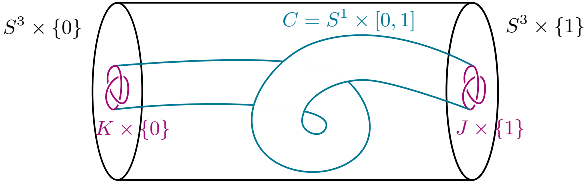

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Theorem (Alexander 1923)

Every knot can be represented as the closure of an n*-braid for some* $n \geq 2$.

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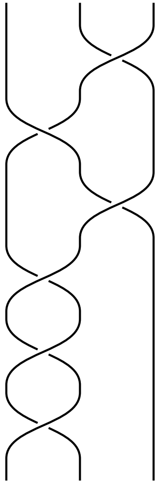
An **n-braid** is a collection of n non-intersecting, never-returning paths in 3-space connecting n points to other n points.



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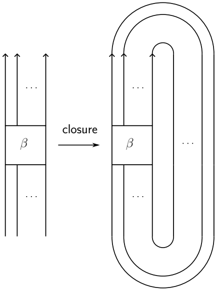

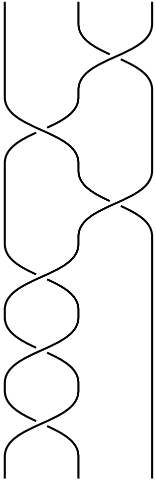

Figure: A 3-braid.

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Isotopy classes of n-braids form the **braid group** B_n on n strands with presentation

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{and} \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| \geq 2 \rangle \qquad \text{(Artin 1925).}$$

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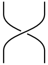

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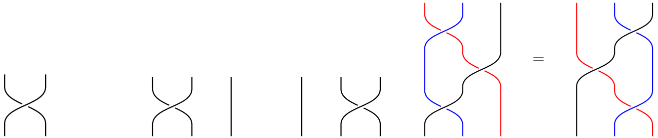

(a) The generator σ_1 of $B_2 = \langle \sigma_1 \rangle$.

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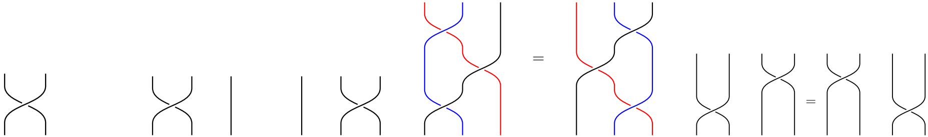

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Strongly quasipositive knots

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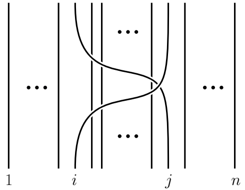

Figure: The positive band word $\sigma_{i,j}$.

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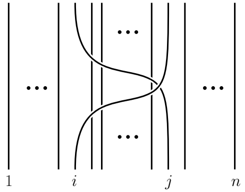

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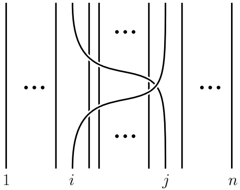

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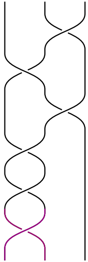
Strongly quasipositive knots

Example

The 3-braid $\beta = \sigma_1 \sigma_1 \underbrace{\sigma_1 \sigma_2 \sigma_1^{-1}}_{=\sigma_{1,3}} \sigma_2$ is strongly quasipositive.



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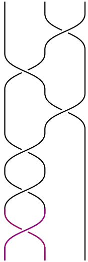

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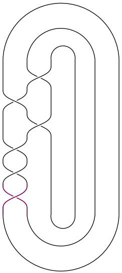

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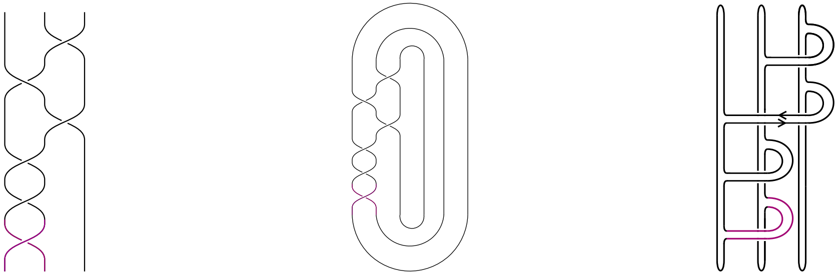

(a) The 3-braid β. (b) Its closure. (c) The surface $F(\beta)$.

Figure: Each strongly quasipositive braid β has an associated canonical Seifert surface $F(\beta)$.

The slice-ribbon conjecture and a conjecture by Baker

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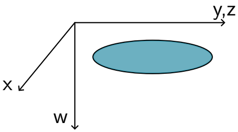
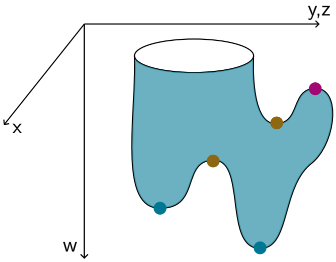
A knot is **slice** if it bounds a smoothly embedded disk in B^4. A knot is **ribbon** if the disk has only local minima and saddles.





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Remark (Hedden)

Baker's conjecture is not true without the fiberedness assumption.

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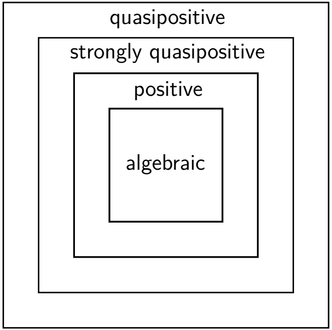
Reformulation of Theorem A:

Every concordance class in $\mathcal{C}$ of a non-trivial strongly quasipositive knot contains **infinitely** many strongly quasipositive knots.

More context on Theorem A



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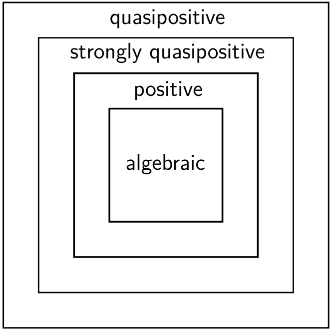

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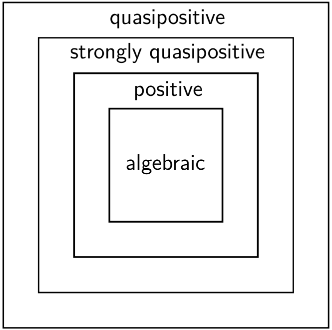

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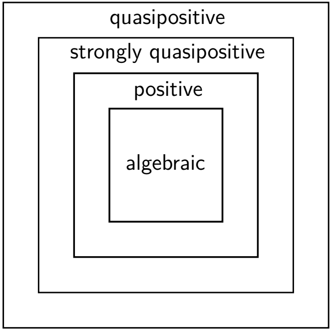

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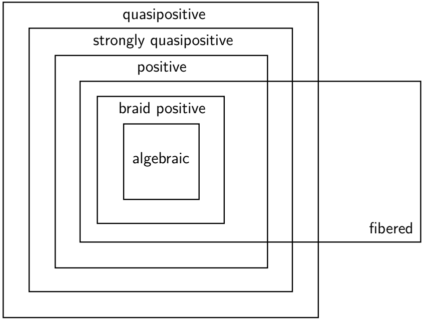

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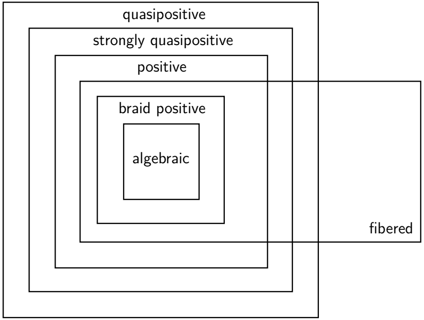

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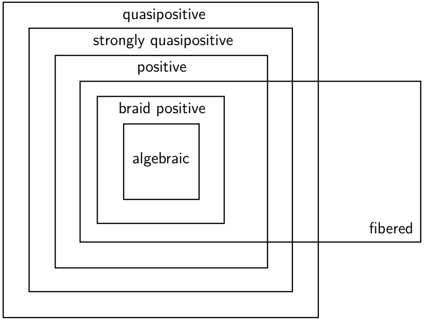

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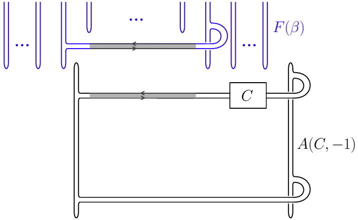

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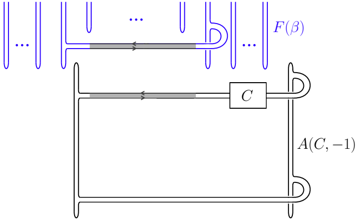

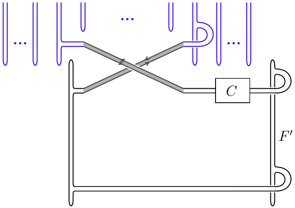

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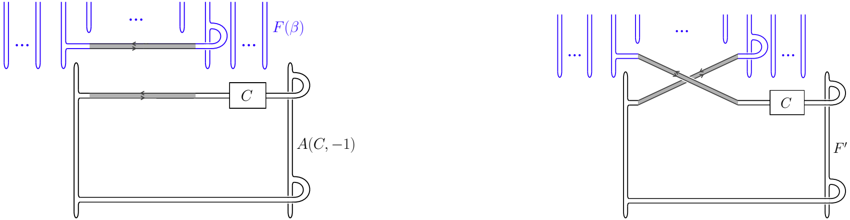

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Claim: $\partial F'$ is a strongly quasipositive knot that is concordant, but not isotopic to K.

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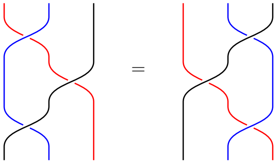

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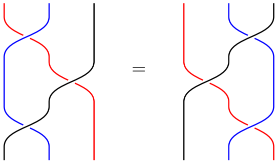

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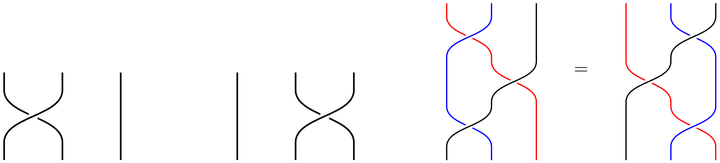

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If K and J are concordant positive 3-braid knots, then this minimal r is the same for both K and J.

On the concordance of positive 3-braid knots

Corollary C (T. 2021)

Let K be a positive 3-braid knot. Then the minimal r such that K is the closure of

$$\alpha = \sigma_1^{p_1}\sigma_2^{q_1}\sigma_1^{p_2}\sigma_2^{q_2}\cdots\sigma_1^{p_r}\sigma_2^{q_r}$$

for integers $r, p_i, q_i \geq 1$ is $r = g(K) + \upsilon(K) + 1$.
If K and J are concordant positive 3-braid knots, then this minimal r is the same for both K and J.
Here, $g(K) = \min\{g(F) \mid F \hookrightarrow S^3$ with or. boundary $\partial F = K\}$.

Question

Is there at most one (braid) positive knot in each concordance class?

More pictures for sketch of proof of Theorem A – 1

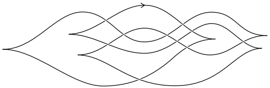


Figure: Front projection of Legendrian representative of $C = \mathrm{m}\,(9_{46})$ with $\mathrm{TB}(C) = -1$.

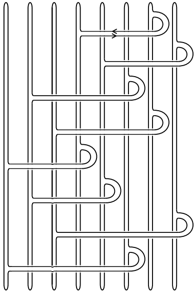
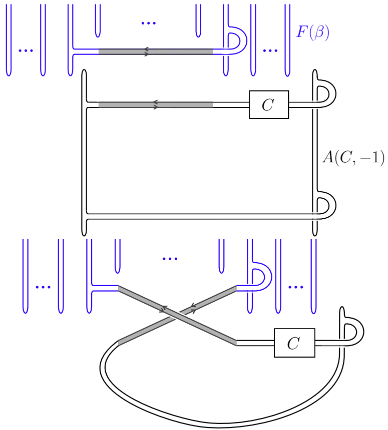


Figure: Strongly quasipositive annulus $A(C, -1)$ for $C = \mathrm{m}\,(9_{46})$.

More pictures for sketch of proof of Theorem A – 2



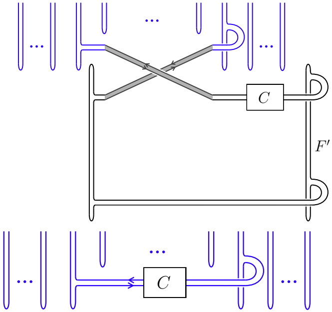


Figure: The surface F' is obtained from $F(\beta)$ by tying the knot C into a band B_β corresponding to the positive band word σ_{i_1,j_1} of β.

More pictures for sketch of proof of Theorem A – 3

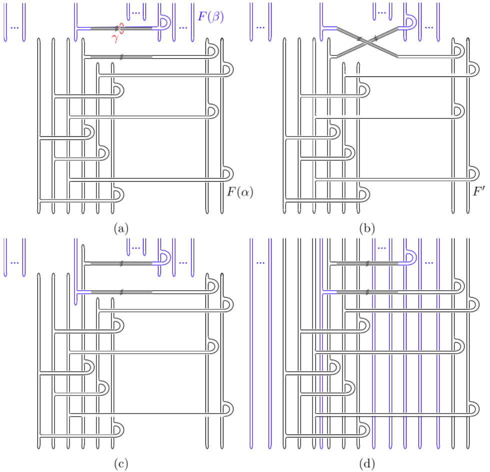


Figure: Quasipositivity of the surface F'.

Picture for sketch of proof of Corollary C

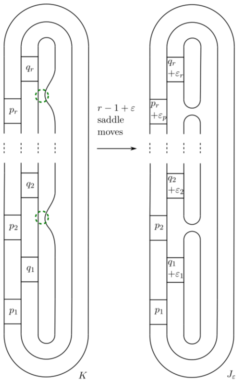


Figure: Schematic of a cobordism between knots $K = \widehat{\beta}$ for $\beta = a^{p_1}b^{q_1}\cdots a^{p_r}b^{q_r}, r, p_i, q_i \geq 1$, $i \in \{1,\dots,r\}$ and $J_\varepsilon = T_{2,\sum_{i=1}^r p_i+\varepsilon_p} \# T_{2,q_1+\varepsilon_1} \# T_{2,q_2+\varepsilon_2} \# \ldots \# T_{2,q_r+\varepsilon_r}$ realized by $r-1+\varepsilon$ saddle moves. This shows $\upsilon(K) \leq -g(K) + r - 1$.

Topoisomerases - 1

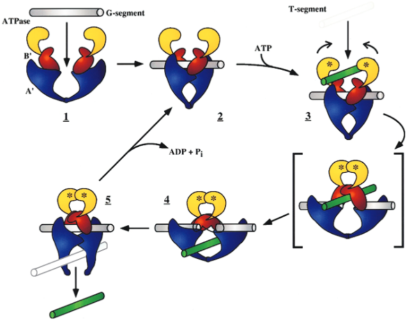
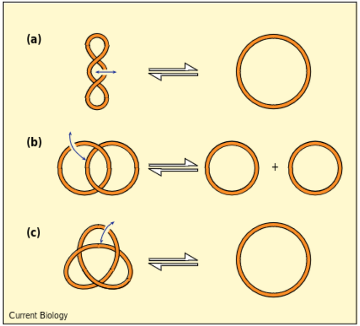


FIGURE 5. Rough mechanism of a type II topoisomerase. Reprinted by permission from Macmillan Publishers Ltd.: *Nature*, Structure and mechanism of DNA topoisomerase II. J. M. Berger, S. J. Gamblin, S. C. Harrison, J. C. Wang (1996).

Figure 1



The reactions carried out by type II topoisomerases: (a) supercoiling/relaxation; (b) catenation/decatenation; (c) knotting/unknotting. All these transformations can be performed by the passage of one double-stranded DNA segment through another (double-headed arrows).

Topoisomerases - 2

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Duplex DNA Knots Produced by *Escherichia coli* Topoisomerase I

STRUCTURE AND REQUIREMENTS FOR FORMATION*

(Received for publication, August 27, 1984)

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We investigated systematically the knotting of nicked circular duplex DNA by *Escherichia coli* topoisomerase I. Agarose gel electrophoresis of knots forms a ladder of DNA bands. Each rung is made up of a variety of knots with the same number of nodes, or segment crossings; knots in adjacent rungs differ by one node. We extended the technique of electron microscopy of *recA* protein-coated DNA to the visualization of the complex knots tied by topoisomerase I. The striking result is that the enzyme produces every knot theoretically possible. The requirement for excess enzyme to form complex knots suggests a role for topoisomerase I in contorting the DNA in addition to promoting strand passage. We conclude that nodes formed are equally likely to be positive or negative and that topoisomerase I can pass DNA strands through a transient enzyme-generated break without regard to orientation of the passing strand. The results are interpreted in terms of a formulation for the topological requirements for knotting.

knots could be untied by enzymes isolated from bacteria, *Drosophila*, *Xenopus*, and mammalian tissues but not by the enzymes, such as *E. coli* topoisomerase I, that change linking number in increments of one. It was concluded that there are two classes of topoisomerases widely distributed in nature, type 1, in which the enzymes act via reversible single-strand breaks, and type 2, acting via double-strand breaks (6).

Since type 1 topoisomerases make transient single-strand breaks in DNA and steps-of-one changes in linking number, it was surprising that these enzymes could also alter the knot and catenane structure of duplex DNA (7, 8). The key to the resolution of this conundrum was the finding of a critical role for pre-existing nicks in the DNA (7), and the analysis of this role is the subject of the companion paper (9). These results clarified the relationship between the prokaryotic type 1 and type 2 mechanisms. Both types of enzyme act at intersections of DNA segments termed nodes. Type 2 enzymes pass one duplex segment through a transient enzyme-bridged break in another in a process called sign inversion. In contrast, type 1 enzymes pass either a duplex or single-stranded segment

3-braid knots with maximal 4-genus – 1

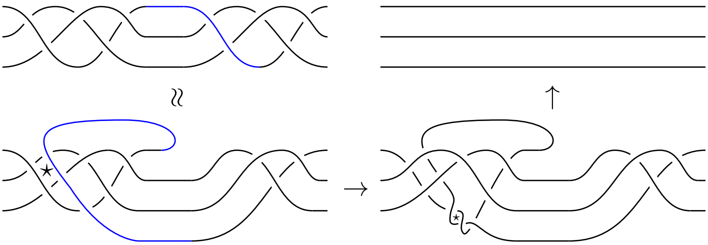

Figure: $abxabx \rightarrowtail \varnothing$ using one twist on four strands, followed by another twist on two strands, at the locations marked $\star$.

3-braid knots with maximal 4-genus – 2

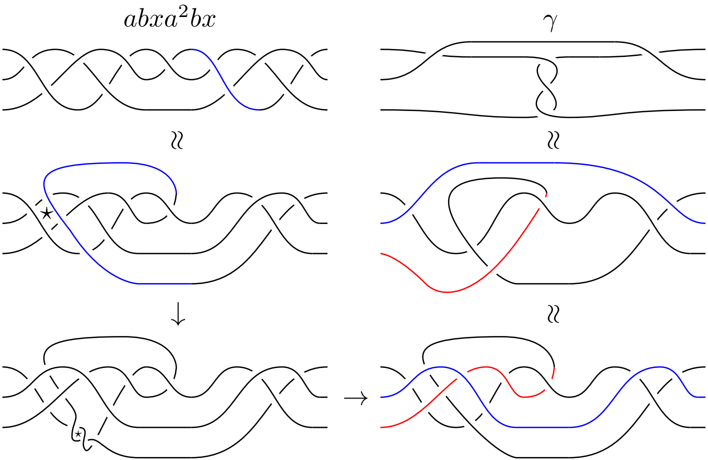

Figure: How to turn the braid $abxa^2bx$ (top left) into the tangle γ (top right) using one twist on four strands, followed by another twist on two strands, at the locations marked $\star$.

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- Slide 7: edited from https://freesvg.org/celtic-knots-design or created by me
- Slide 10: Conway knot from https://knotinfo.math.indiana.edu/ [LM23]

Other figures created by me with inspiration from:

- Daniele Celoria for schematic of concordance (slide 11)
- Arunima Ray for schematics of sliceness (slides 10 and 21)

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