ETH zürich

D MATH

On notions of braid positivity and knot concordance

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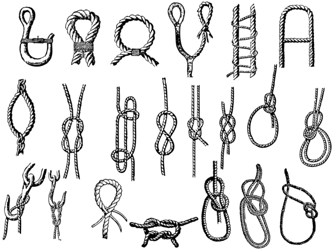
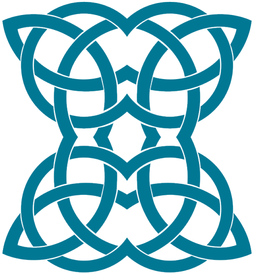
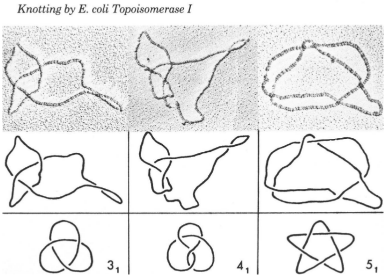
Outline

1. Introduction
   - Introduction to knots
   - Slice knots and knot concordance
   - Knots as closures of braids
2. Strongly quasipositive knots are concordant to infinitely many such knots
   - Strongly quasipositive knots and a conjecture by Baker
   - Theorem A: Statement and more context
   - Sketch of proof
3. On the concordance of positive 3-braid knots

Examples of knots







Dean–Stasiak–Koller–Cozzarelli 1985

Shakespeare, ∼1600

O time, thou must untangle this, not I.

It is too hard a knot for me t'untie.

What is a mathematical knot?

Start with a piece of string, tie a knot in it, and glue the two ends together.

Definition

A **knot** is a simple closed curve in space.

What is a mathematical knot?



Start with a piece of string, tie a knot in it, and glue the two ends together.

Definition

A **knot** is a **simple** closed curve in space, i. e. a curve with **no self-intersections**.

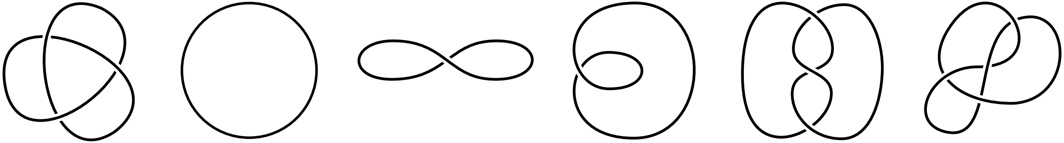
What is a mathematical knot?

Start with a piece of string, tie a knot in it, and glue the two ends together.

Definition

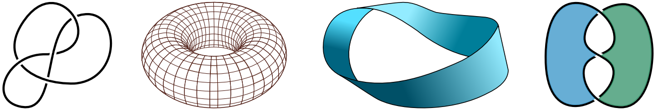
A **knot** is a simple closed curve in space.

Some examples of (mathematical) knots:

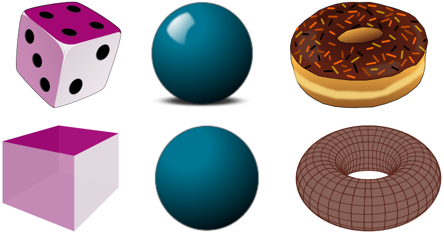


The second one is the **unknot**.

Low-dimensional topology and knot theory



Topology studies properties of spaces that are preserved under continuous deformations.
Low-dimensional topology focuses on spaces of dimension 4 and below.

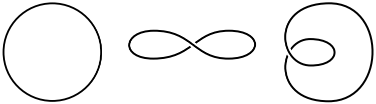


We can continuously deform a cube into a ball. But we cannot continuously deform a ball into a torus.
Knot theory studies knots in 3-space and their properties preserved under **continuous deformations**.

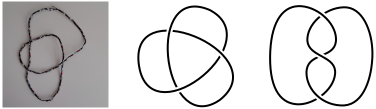
Knot theory as a subarea of low-dimensional topology

Definition

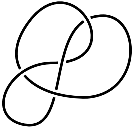
Two knots are **isotopic** if there is a continuous deformation of one knot into the other, i. e. without cutting the piece of string or passing it through itself.



(a) The unknot.



(b) The trefoil knot.



(c) The figure-eight knot.

Figure: Examples of **(isotopy classes of)** knots.

Why (k)not?

Motto: Knots help to understand 3- and 4-dimensional manifolds.

1) Knots provide **blueprints** for constructing 3-manifolds.

Theorem (Lickorish, Wallace 1960s)

*Any oriented, closed, connected **3-manifold** can be obtained from the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$ by performing **Dehn surgery** on a collection of knots.*

2) Knots can be used to reveal **exotic** structures of 4-manifolds.

Theorem (Moise, Stallings, Taubes, Gompf, Freedman 1960s–1980s)

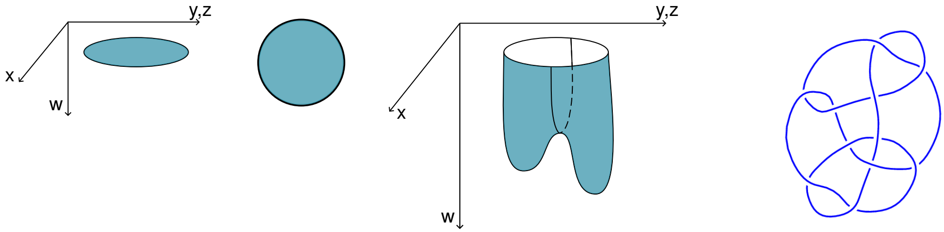
*For $n \neq 4$, there is a unique smooth structure on $\mathbb{R}^n$. In contrast, there are **uncountably many smooth structures** on $\mathbb{R}^4$. This can be shown using the existence of topologically, but not smoothly **slice** knots.*

Slice knots and knot concordance

Motto: A slice knot is the next best thing to an unknot in 4 dimensions.

Definition

A knot in S^3 is **slice** if it bounds a smoothly embedded **disk** in B^4, the 4-ball bounded by S^3.



Proposition

A knot is isotopic to the unknot if and only if it bounds a disk in S^3.

Theorem (Piccirillo 2020)

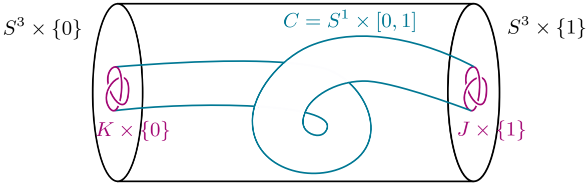
The Conway knot is not slice.

Slice knots and knot concordance

Motto: Concordance generalizes isotopy between knots to dimension 4.

Definition

Two knots K and J in S^3 are **concordant** if they cobound a smoothly embedded **cylinder** in $S^3 \times [0,1]$.



Concordance is an **equivalence relation**. The **concordance group** is the (countable abelian) group

$$\mathcal{C} = (\{\text{concordance classes of knots}\}, \text{connected sum}) \qquad \text{(Fox–Milnor 1966).}$$

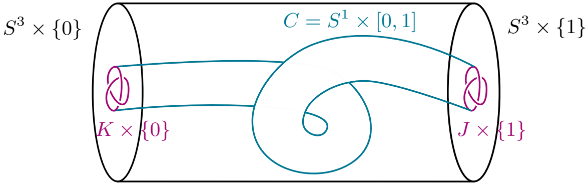
Slice knots are concordant to the unknot and represent the identity element.

Slice knots and knot concordance

Motto: Concordance generalizes isotopy between knots to dimension 4.

Definition

Two knots K and J in S^3 are **concordant** if they cobound a smoothly embedded **cylinder** in $S^3 \times [0,1]$.



Proposition

Let K and J be knots in S^3. Then:

- K and J are isotopic knots. $\Rightarrow$ K and J are concordant.
- K and J are isotopic knots. $\not\Leftarrow$ K and J are concordant. Example: Any nontrivial slice knot.

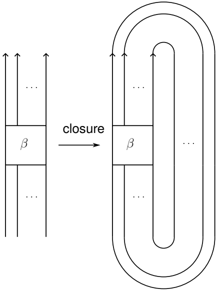
Knots as closures of braids

Theorem (Alexander 1923)

Every knot can be represented as the closure of an n-braid for some $n \geq 2$.

Definition

An **n-braid** is a collection of n non-intersecting, never-returning paths in 3-space connecting n points to other n points.



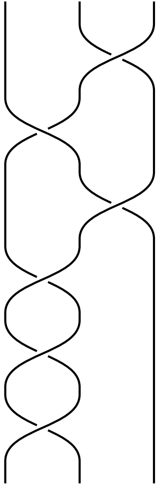
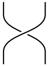


Figure: A 3-braid.

Knots as closures of braids

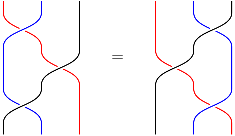
Isotopy classes of n-braids form the **braid group** B_n on n strands with presentation

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{and} \quad \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } |i-j| \geq 2\rangle \qquad \text{(Artin 1925).}$$

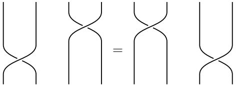


(a) The generator σ_1 of $B_2 = \langle \sigma_1 \rangle$.

(b) The generators σ_1 and σ_2 of B_3.



(c) The braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ in B_3.



(d) The relation $\sigma_1\sigma_3 = \sigma_3\sigma_1$ in B_4.

Strongly quasipositive knots

Definition (Rudolph)

An n-braid β is **strongly quasipositive** if it is a product of certain conjugates of the positive Artin generators σ_j of B_n, namely the **positive band words** $\sigma_{i,j}$, where

$$\sigma_{i,j} = (\sigma_i \cdots \sigma_{j-2})\, \sigma_{j-1}\, (\sigma_i \cdots \sigma_{j-2})^{-1} \; (= \omega_{i,j}\sigma_{j-1}\omega_{i,j}^{-1}) \qquad \text{for} \quad 1 \leq i < j \leq n,$$

i. e. $$\beta = \prod_{k=1}^{m} \sigma_{i_k,j_k} \qquad \text{for} \quad 1 \leq i_k < j_k \leq n,\ 1 \leq k \leq m.$$

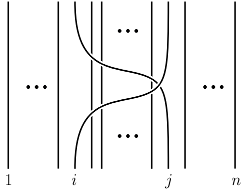


Figure: The positive band word $\sigma_{i,j}$.

Strongly quasipositive knots

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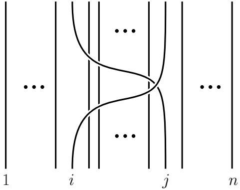


Figure: The positive band word $\sigma_{i,j}$.

Strongly quasipositive knots

Definition (Rudolph)

An n-braid β is **strongly quasipositive** if it is a finite product of certain conjugates of the positive Artin generators σ_i of B_n, namely the **positive band words** $\sigma_{i,j}$, where

$$\sigma_{i,j} = (\sigma_i \cdots \sigma_{j-2})\, \sigma_{j-1}\, (\sigma_i \cdots \sigma_{j-2})^{-1} \qquad \text{for} \quad 1 \leq i < j \leq n,$$

i. e. $$\beta = \prod_{k=1}^{m} \sigma_{i_k, j_k} \qquad \text{for} \quad 1 \leq i_k < j_k \leq n,\ 1 \leq k \leq m.$$

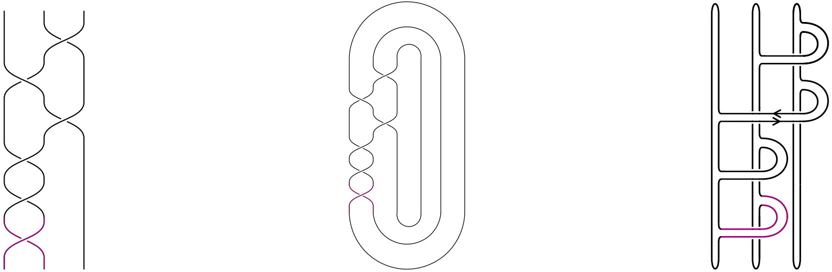
Definition (Rudolph)

A knot is **strongly quasipositive** if it is the closure of a strongly quasipositive n-braid for some $n \geq 2$.

Strongly quasipositive knots

Example

The 3-braid $\beta = \sigma_1\sigma_1 \underbrace{\sigma_1\sigma_2\sigma_1^{-1}}_{=\sigma_{1,3}} \sigma_2$ is strongly quasipositive. Its closure is a strongly quasipositive knot.



(a) The 3-braid β. (b) Its closure. (c) The surface $F(\beta)$.

Figure: Each strongly quasipositive braid β has an associated canonical Seifert surface $F(\beta)$.

The slice-ribbon conjecture and a conjecture by Baker

Recall that a knot is **strongly quasipositive** if it is the closure of a strongly quasipositive n-braid.

Definition

A knot K in S^3 is **fibered** if there exists a locally trivial fiber bundle $S^3 \setminus K \to S^1$ whose fibers are the interiors of Seifert surfaces for the knot.

Conjecture (Baker 2016)

If two strongly quasipositive, fibered knots are concordant, then they are isotopic.

Theorem (Baker 2016)

Slice-ribbon conjecture $\Rightarrow$ *Baker's conjecture.*

The slice-ribbon conjecture and a conjecture by Baker

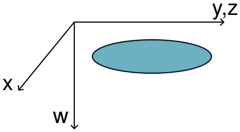
Theorem (Baker 2016)

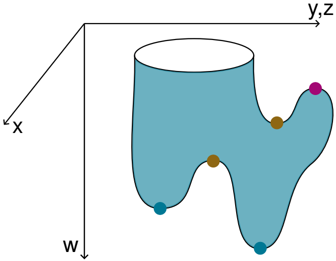
Slice-ribbon conjecture $\Rightarrow$ Baker's conjecture.

Slice-ribbon conjecture (Fox 1962)

Every slice knot is ribbon.

A knot is **slice** if it bounds a smoothly embedded disk in B^4. A knot is **ribbon** if the disk has only local minima and saddles.





The slice-ribbon conjecture and a conjecture by Baker

Conjecture (Baker 2016)

If two strongly quasipositive, fibered knots are concordant, then they are isotopic.

Theorem (Baker 2016)

Slice-ribbon conjecture $\Rightarrow$ *Baker's conjecture.*

Slice-ribbon conjecture (Fox 1962)

Every slice knot is ribbon.

Remark (Hedden)

Baker's conjecture is not true without the fiberedness assumption.

Theorem A

Conjecture (Baker 2016)

If two strongly quasipositive, fibered knots are concordant, then they are isotopic.

Theorem A (T. 2022)

Every non-trivial strongly quasipositive knot is concordant to infinitely many pairwise non-isotopic strongly quasipositive knots.

Remark

There is only one strongly quasipositive, slice knot: the **unknot**, since for strongly quasipositive knots, the genus and the smooth 4-genus coincide (Bennequin 1983, Rudolph 1993).

Reformulation of Theorem A:

Every concordance class in $\mathcal{C}$ of a non-trivial strongly quasipositive knot contains **infinitely** many strongly quasipositive knots.

More context on Theorem A

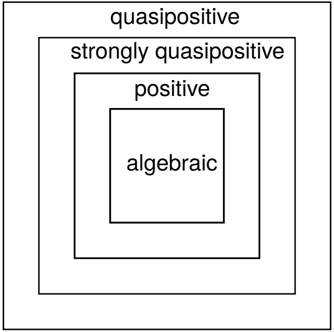


Figure: Notions of positivity.

Theorem A (T. 2022)

Every concordance class in $\mathcal{C}$ of a non-trivial strongly quasipositive knot contains **infinitely** many strongly quasipositive knots.

Theorem (Baader–Dehornoy–Liechti 2017)

Every concordance class in $\mathcal{C}$ contains at most ***finitely*** *many positive knots.*

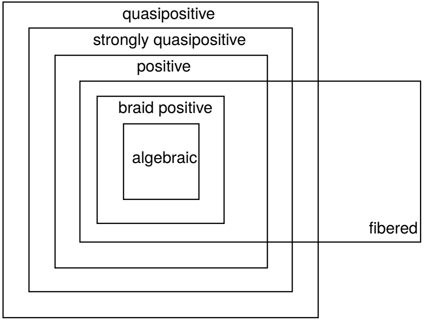
Theorem (Litherland 1979)

Every concordance class in $\mathcal{C}$ contains at most ***one*** *algebraic knot.*

Example: Torus knots

$$T_{p,q} = \widehat{(\sigma_1\sigma_2\ldots\sigma_{p-1})^q}$$
$$= V_f \cap S^3 \subseteq \mathbb{C}^2 \qquad \text{for} \quad f(x,y) = x^p - y^q \in \mathbb{C}[x,y].$$

More context on Theorem A



Question

Are there only finitely many strongly quasipositive, fibered knots in each concordance class?

Question

Is there at most one (braid) positive knot in each concordance class?

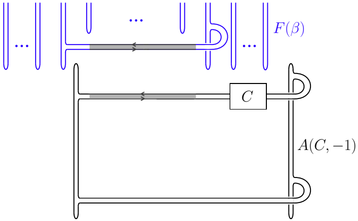
Idea of the proof

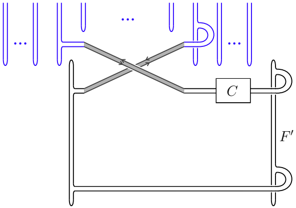
Theorem A (T. 2022)

Every non-trivial strongly quasipositive knot is concordant to infinitely many pairwise non-isotopic strongly quasipositive knots.

Idea of the proof:

Let $K = \partial F(\beta)$ be a non-trivial knot for a strongly quasipositive braid $\beta = \prod_{k=1}^{m} \sigma_{i_k, j_k}$. Take a nontrivial slice knot C with $\mathrm{TB}(C) = -1$, e.g. $C = \mathrm{m}\,(9_{46})$.





The surface F' is obtained from $F(\beta)$ by tying the knot C into the band $B_{\sigma_{i_1, j_1}}$ of $F(\beta)$.
Claim: $\partial F'$ is a strongly quasipositive knot that is concordant, but not isotopic to K.

On the concordance of positive 3-braid knots

Question

Is there at most one (braid) positive knot in each concordance class?

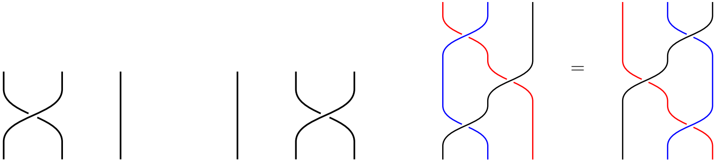
We focus on braids on 3 strands.

Definition

A 3-braid β is **positive** if $\beta = \sigma_{i_1} \cdots \sigma_{i_m}$ for some $i_1, \cdots, i_m \in \{1, \ldots, n-1\}$ (no inverses σ_j^{-1}).

Definition

A knot is a **positive 3-braid knot** if it is the closure of a positive 3-braid.



(a) Generators σ_1 and σ_2 of B_3.

(b) Braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

On the concordance of positive 3-braid knots

Let $\upsilon\colon \mathcal{C} \to \mathbb{Z}$ denote the smooth concordance invariant **upsilon** from knot Floer homology (Ozsváth–Stipsicz–Szabó 2017) such that

- $\upsilon\colon \mathcal{C} \to \mathbb{Z}$ is a group homomorphism, i. e. $\upsilon(K\#J) = \upsilon(K) + \upsilon(J)$ for all knots K and J,
- $|\upsilon(K)| \leq g_4(K) = \min\{g(F) \mid F \hookrightarrow B^4$ with or. boundary $\partial F = K$ in $S^3 = \partial B^4\}$.

Theorem B (T. 2021)

Let $\beta = \Delta^{2\ell}\sigma_1^{-p_1}\sigma_2^{q_1}\sigma_1^{-p_2}\sigma_2^{q_2}\cdots\sigma_1^{-p_r}\sigma_2^{q_r} \in B_3$ for some $\ell \in \mathbb{Z}$, $r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \ldots, r\}$, where $\Delta^2 = (\sigma_1\sigma_2)^3$. Suppose that $K = \widehat{\beta}$ is a knot. Then

$$\upsilon(K) = \frac{\sum_{i=1}^{r}(p_i - q_i)}{2} - 2\ell.$$

On the concordance of positive 3-braid knots

Theorem B (T. 2021)

Let $\beta = \Delta^{2\ell}\sigma_1^{-p_1}\sigma_2^{q_1}\sigma_1^{-p_2}\sigma_2^{q_2}\cdots\sigma_1^{-p_r}\sigma_2^{q_r} \in B_3$ for some $\ell \in \mathbb{Z}$, $r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \ldots, r\}$, where $\Delta^2 = (\sigma_1\sigma_2)^3$. Suppose that $K = \widehat{\beta}$ is a knot. Then

$$\upsilon(K) = \frac{\sum_{i=1}^{r}(p_i - q_i)}{2} - 2\ell.$$

Corollary C (T. 2021)

Let K be a positive 3-braid knot. Then the minimal r such that K is the closure of

$$\alpha = \sigma_1^{p_1}\sigma_2^{q_1}\sigma_1^{p_2}\sigma_2^{q_2}\cdots\sigma_1^{p_r}\sigma_2^{q_r}$$

for integers $r, p_i, q_i \geq 1$ is $r = g(K) + \upsilon(K) + 1$.
Here, $g(K) = \min\{g(F) \mid F \hookrightarrow S^3$ with or. boundary $\partial F = K\}$.
If K and J are concordant positive 3-braid knots, then this minimal r is the same for both K and J.

On the concordance of positive 3-braid knots

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If K and J are concordant positive 3-braid knots, then this minimal r is the same for both K and J.
Here, $g(K) = \min\{g(F) \mid F \hookrightarrow S^3$ with or. boundary $\partial F = K\}$.

Question

Is there at most one (braid) positive knot in each concordance class?

More pictures for sketch of proof of Theorem A – 1

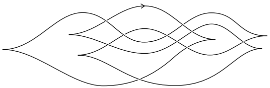


Figure: Front projection of Legendrian representative of $C = \mathrm{m}\,(9_{46})$ with $\mathrm{TB}(C) = -1$.

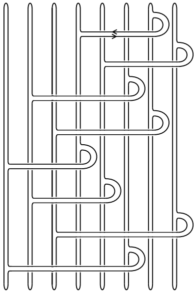


Figure: Strongly quasipositive annulus $A(C, -1)$ for $C = \mathrm{m}\,(9_{46})$.

More pictures for sketch of proof of Theorem A – 2

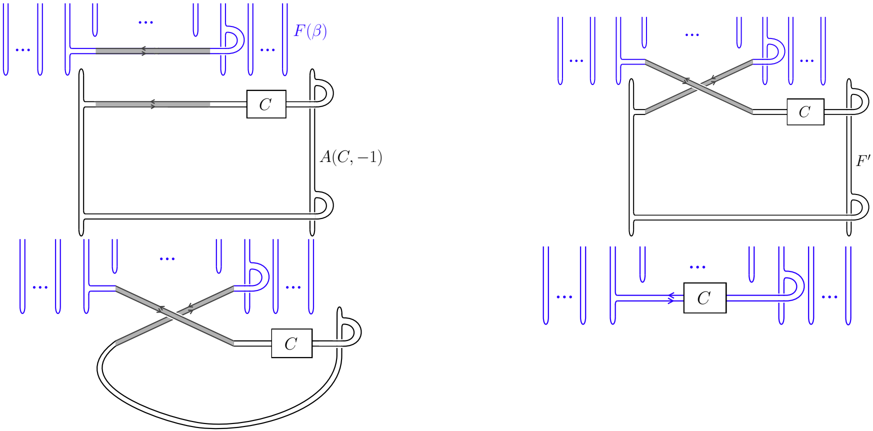


Figure: The surface F' is obtained from $F(\beta)$ by tying the knot C into a band B_β corresponding to the positive band word σ_{i_1,j_1} of β.

More pictures for sketch of proof of Theorem A – 3

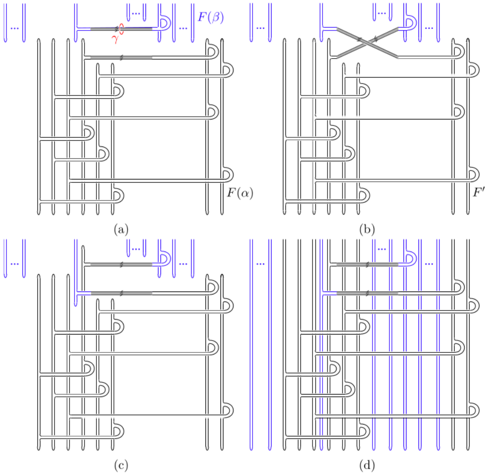


Figure: Quasipositivity of the surface F'.

Picture for sketch of proof of Corollary C

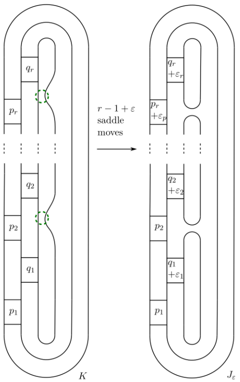


Figure: Schematic of a cobordism between knots $K = \widehat{\beta}$ for $\beta = a^{p_1}b^{q_1}\cdots a^{p_r}b^{q_r}, r, p_i, q_i \geq 1$, $i \in \{1,\dots,r\}$ and $J_\varepsilon = T_{2,\sum_{i=1}^r p_i+\varepsilon_p} \# T_{2,q_1+\varepsilon_1} \# T_{2,q_2+\varepsilon_2} \# \dots \# T_{2,q_r+\varepsilon_r}$ realized by $r-1+\varepsilon$ saddle moves. This shows $\upsilon(K) \leq -g(K)+r-1$.

Topoisomerases - 1

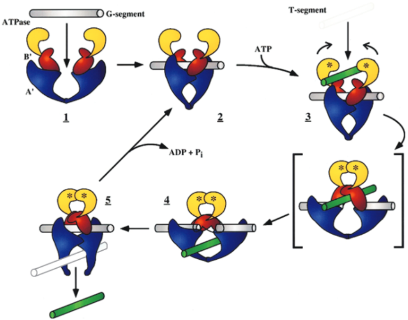
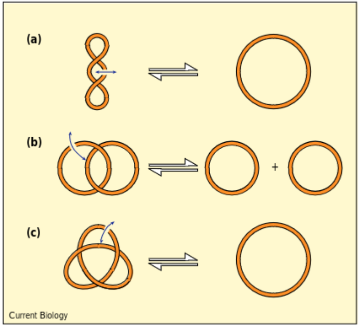


FIGURE 5. Rough mechanism of a type II topoisomerase. Reprinted by permission from Macmillan Publishers Ltd.: *Nature*, Structure and mechanism of DNA topoisomerase II. J. M. Berger, S. J. Gamblin, S. C. Harrison, J. C. Wang (1996).

Figure 1



The reactions carried out by type II topoisomerases: (a) supercoiling/relaxation; (b) catenation/decatenation; (c) knotting/unknotting. All these transformations can be performed by the passage of one double-stranded DNA segment through another (double-headed arrows).

Topoisomerases - 2

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Duplex DNA Knots Produced by *Escherichia coli* Topoisomerase I

STRUCTURE AND REQUIREMENTS FOR FORMATION*

(Received for publication, August 27, 1984)

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We investigated systematically the knotting of nicked circular duplex DNA by *Escherichia coli* topoisomerase I. Agarose gel electrophoresis of knots forms a ladder of DNA bands. Each rung is made up of a variety of knots with the same number of nodes, or segment crossings; knots in adjacent rungs differ by one node. We extended the technique of electron microscopy of *recA* protein-coated DNA to the visualization of the complex knots tied by topoisomerase I. The striking result is that the enzyme produces every knot theoretically possible. The requirement for excess enzyme to form complex knots suggests a role for topoisomerase I in contorting the DNA in addition to promoting strand passage. We conclude that nodes formed are equally likely to be positive or negative and that topoisomerase I can pass DNA strands through a transient enzyme-generated break without regard to orientation of the passing strand. The results are interpreted in terms of a formulation for the topological requirements for knotting.

knots could be untied by enzymes isolated from bacteria, *Drosophila*, *Xenopus*, and mammalian tissues but not by the enzymes, such as *E. coli* topoisomerase I, that change linking number in increments of one. It was concluded that there are two classes of topoisomerases widely distributed in nature, type 1, in which the enzymes act via reversible single-strand breaks, and type 2, acting via double-strand breaks (6).

Since type 1 topoisomerases make transient single-strand breaks in DNA and steps-of-one changes in linking number, it was surprising that these enzymes could also alter the knot and catenane structure of duplex DNA (7, 8). The key to the resolution of this conundrum was the finding of a critical role for pre-existing nicks in the DNA (7), and the analysis of this role is the subject of the companion paper (9). These results clarified the relationship between the prokaryotic type 1 and type 2 mechanisms. Both types of enzyme act at intersections of DNA segments termed nodes. Type 2 enzymes pass one duplex segment through a transient enzyme-bridged break in another in a process called sign inversion. In contrast, type 1 enzymes pass either a duplex or single-stranded segment

Topoisomerases - 3

4978 *Knotting by E. coli Topoisomerase I*

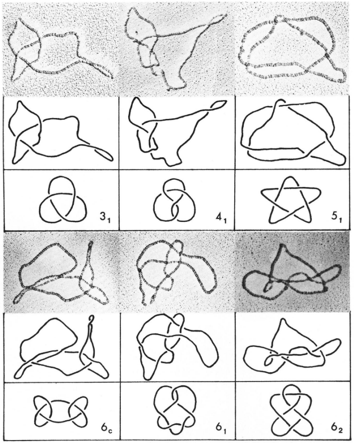

FIG. 3. **Electron micrographs of DNA knots.** Purified knot preparations described in Fig. 2 were coated with *recA* protein, mounted, and photographed in the electron microscope. Molecules shown here are magnified approximately × 80,000. *Beneath each micrograph* is a tracing of the knot showing the over- and under-passing strand at each node. *Below* the *tracing* is an idealized *redrawing* of the knot and an *identification number*, consisting of the number of nodes, with a *subscript* indicating the knot's place in a standard tabulation (24, 25). A subscript *c* indicates a composite knot, composed of two distinct prime knots tied into a single ring.

3-braid knots with maximal 4-genus – 1

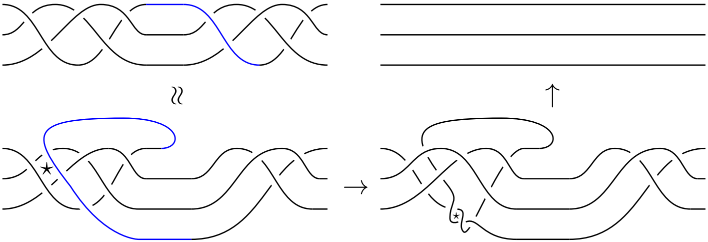

Figure: $abxabx \rightarrowtail \varnothing$ using one twist on four strands, followed by another twist on two strands, at the locations marked $\star$.

3-braid knots with maximal 4-genus – 2

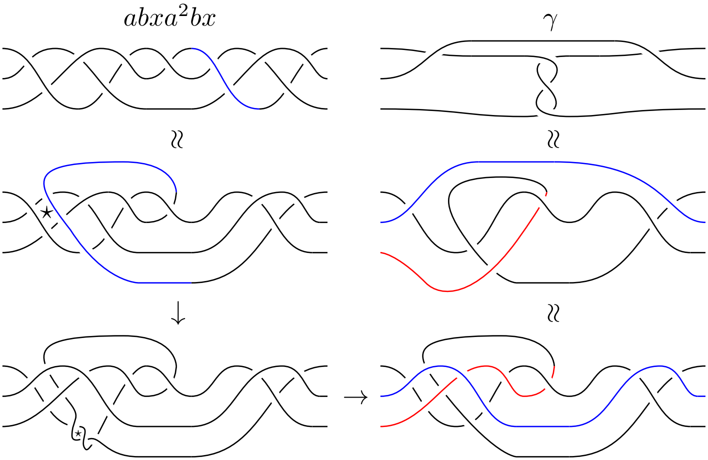

Figure: How to turn the braid $abxa^2bx$ (top left) into the tangle γ (top right) using one twist on four strands, followed by another twist on two strands, at the locations marked $\star$.

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- Slide 7: edited from https://freesvg.org/celtic-knots-design or created by me
- Slide 10: Conway knot from https://knotinfo.math.indiana.edu/ [LM23]

Other figures created by me with inspiration from:

- Daniele Celoria for schematic of concordance (slide 11)
- Arunima Ray for schematics of sliceness (slides 10 and 21)

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