Strongly quasipositive knots are concordant to infinitely many strongly quasipositive knots

CIRGET, December 16, 2022

<ins>Thm</ins> (T. 2022): Every non-trivial strongly quasipositive knot is (smoothly) concordant to infinitely many pairwise non-isotopic strongly quasipositive knots.

In this talk:

I) Knots and concordance

II) Strongly quasipositive knots

III) Context

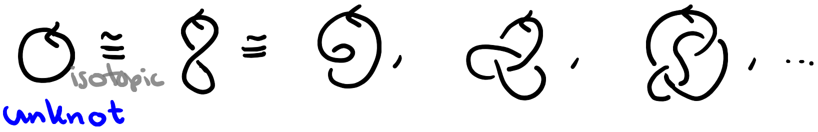
IV) Sketch of pf

Knot concordance

We work in the smooth category.

A **link** is a non-empty, oriented, closed, smooth 1-dimensional submanifold of S^3, up to (ambient) isotopy. A **knot** is a connected link.

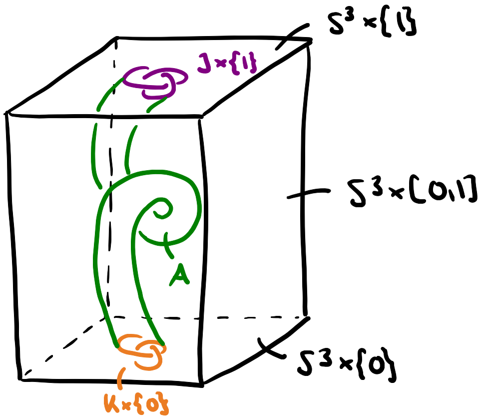
$\overset{\text{isotopy}}{\parallel}$ (or. pres. diffeomorphism of S^3)

Ex: 

Two knots K, J are **concordant** if $\exists\ S^1 \times [0,1] \cong A \subseteq S^3 \times [0,1]$

(A: smoothly & properly embedded oriented annulus)

s.t. $\partial A = K \times \{0\} \cup \overleftarrow{J} \times \{1\}$. ($\overleftarrow{J}$: reverse or.)



This is an equivalence relation and in fact,

$\mathcal{C} := (\{\text{concordance classes}\}, \#)$

is a group.

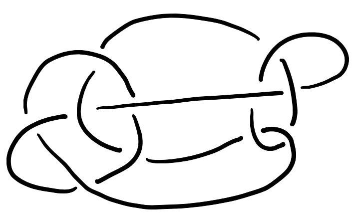
Fact: K, J isotopic knots $\Longrightarrow$ K, J are concordant

In general, $\not\Leftarrow$

Ex: $\exists$ non-trivial (not isotopic to ◯ unknot) knots that are not **slice**.

$\overset{\text{slice}}{\parallel}$ concordant to the unknot

e.g. the Conway knot is not slice. [Piccirillo 2020]



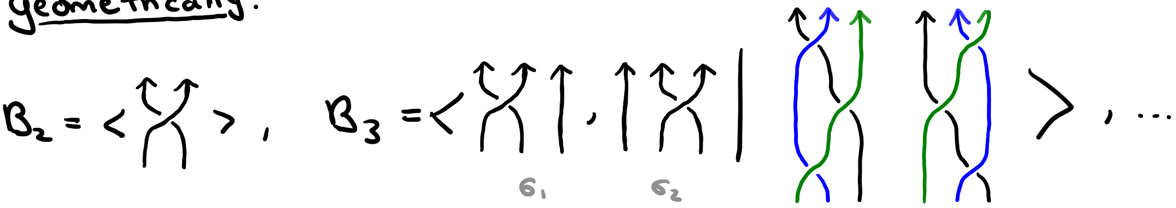
(Strongly) quasipositive knots

Thm (Alexander 1923): Any knot (or link) is the closure of an n-braid for some $n \geq 2$.

Artin 1925:
$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| \geq 2 \end{array} \right\rangle$$

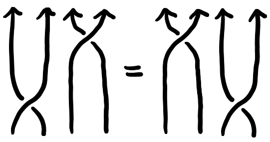
braid gp on n strands

geometrically:

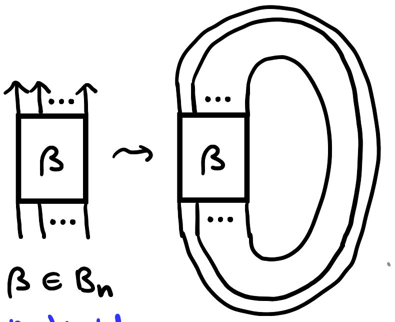


$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$
(braid relation)

in B_4:



$\sigma_1 \sigma_3 = \sigma_3 \sigma_1$



$\beta \in B_n$
n-braid

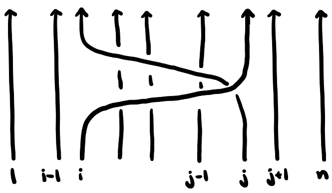
closure $\hat{\beta}$ ⇝ knot or link

A braid $\beta \in B_n$ is quasipositive if $\beta = \prod_{k=1}^{m} w_k \sigma_{i_k} w_k^{-1}$, $i_k \in \{1, \dots, n-1\}$, $w_k \in B_n$, $k = 1, \dots, m$.

A braid $\beta \in B_n$ is strongly quasipositive (sqp) if $\beta = \prod_{k=1}^{m} \sigma_{i_k, j_k}$

for some $1 \leq i_k < j_k \leq n$, $k = 1, \dots, m$, where

$$\sigma_{i,j} = (\sigma_i \cdot \ldots \cdot \sigma_{j-2})\, \sigma_{j-1} (\sigma_i \cdot \ldots \cdot \sigma_{j-2})^{-1} \quad \text{for } i<j$$



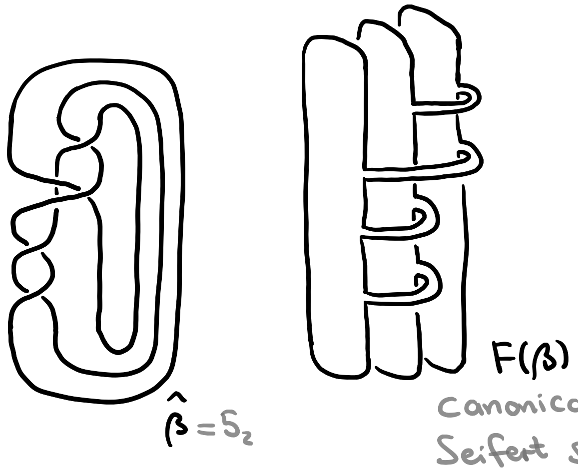
$\sigma_{i,j}$

Note: $\sigma_{i,i+1} = \sigma_i$

(positive band)

A knot or link is (strongly) quasipositive if it is the closure of a (strongly) quasipositive braid.

Ex: $\beta = \sigma_1^2 \underbrace{(\sigma_1 \sigma_2 \sigma_1^{-1})}_{\sigma_{1,3}} \sigma_2 \in B_3$



canonical Seifert surface associated to β

$\chi(F(\beta)) = 3 - 4 = -1$

$g(F(\beta)) = \frac{1 - \chi(F(\beta))}{2} = 1$

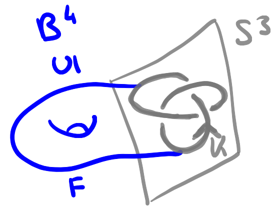
Thm (slice-Bennequin inequality) [Bennequin '83, Rudolph '93, Kronheimer–Mrowka '93]

Let $\beta \in B_n$ be a sqp braid, $K = \hat{\beta}$ a knot.

$$\Rightarrow g(K) = g_4(K) = g(F(\beta)) = \frac{wr(\beta) - n + 1}{2}$$

writhe

where $g_4(K) = \min\{ g(F) \mid F$ or., cpct, conn. surface in S^3 B^4 w/ or. bdry K in $S^3 = \partial B^4 \}$



Thm (T. '22): Every non-trivial strongly quasipositive knot link is concordant to infinitely many pairwise non-isotopic strongly quasipositive knots links.

Rem: Thm not true for $U = \bigcirc$ b/c it is the only strongly quasipositive knot concordant to U.

Context:

knots of isolated singularities of cx. alg. plane curves

$$\left\{\begin{matrix}\text{torus}\\ \text{knots}\\ T_{p,q}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{algebraic}\\ \text{knots}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{positive}\\ \text{knots}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{sqp}\\ \text{knots}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{qp}\\ \text{knots}\end{matrix}\right\} = \left\{\begin{matrix}\text{transverse}\\ \text{intersections of}\\ \text{complex algebraic}\\ \text{plane curves in } \mathbb{C}^2\\ \text{w/ } S^3 \subseteq \mathbb{C}^2\end{matrix}\right\}$$

Nakamura '98
Rudolph '99

Rudolph '98
Boileau-Orevkov '01

$T_{p,q} = (\sigma_1 \cdot \ldots \cdot \sigma_{p-1})^q$

$f(x,y) = x^p + y^q$

$T_{p,q} = \underset{\{f=0\}}{\underset{\shortparallel}{V_f}} \cap S^3 \subseteq \mathbb{C}^2$

Thm (Litherland '79): concordance $\Rightarrow$ isotopy for algebraic knots.

reformulation: Every concordance class in $\mathcal{C}$ contains at most one algebraic knot.

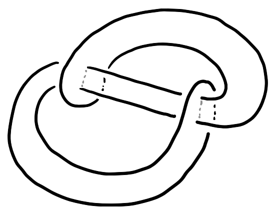
Thm (Baader – Dehornoy - Liechti '17): Every concordance class in $\mathcal{C}$ contains at most finitely many positive knots.

Conj. (Baker '16): concordance $\Rightarrow$ isotopy for sqp, fibered knots

Thm (Baker '16): If the conjecture turns out to be false, then the slice-ribbon conjecture is false.
Fox '62

concordant to ○

slice-ribbon conjecture (Fox '62): Every slice knot is ribbon, i.e. bounds an immersed disk w/ only ribbon singularities.

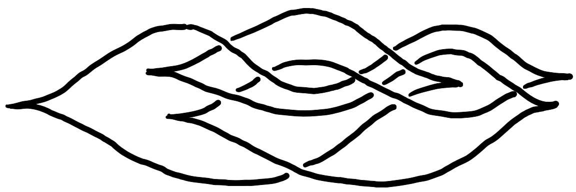


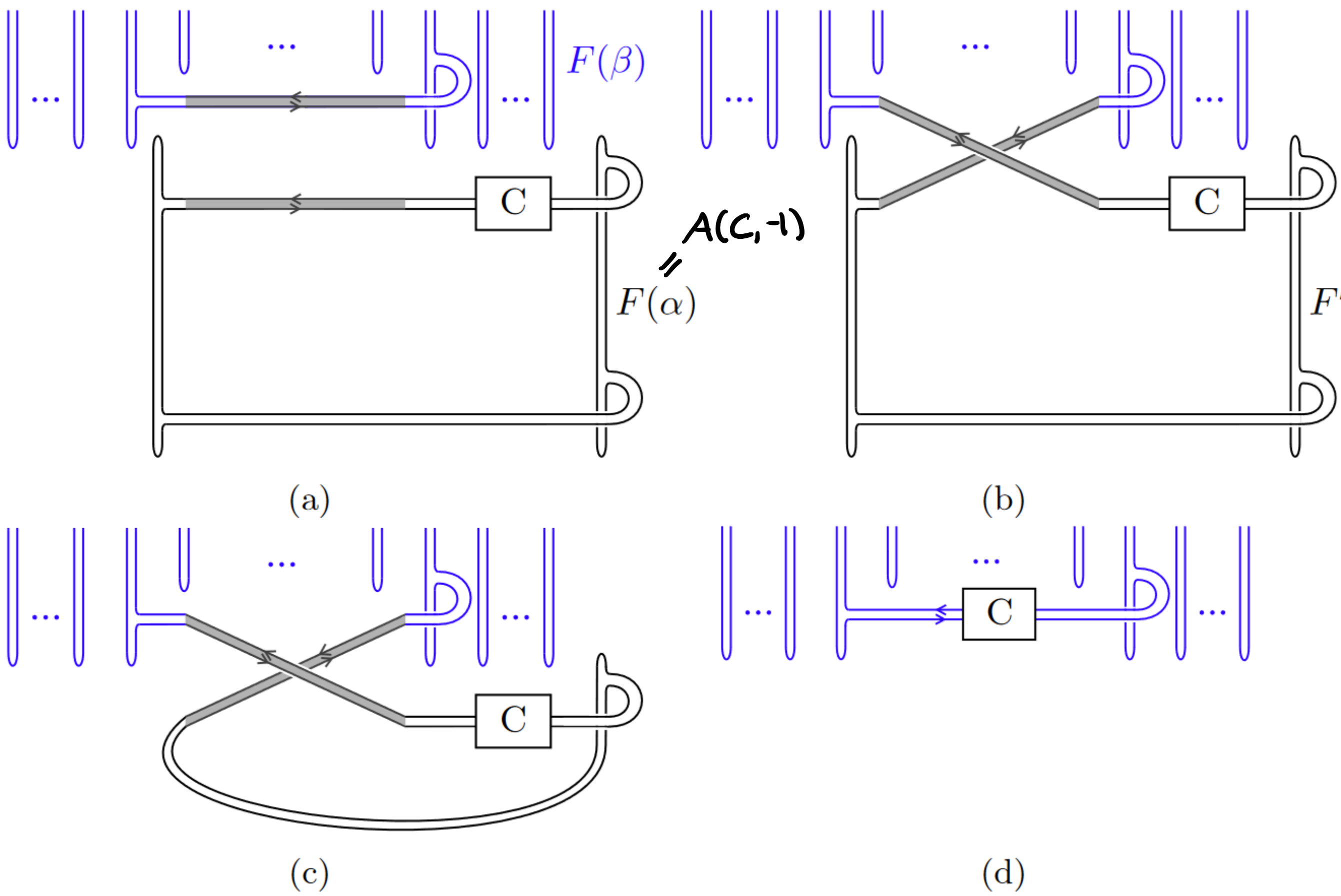
Thm (T. '22): Every non-trivial strongly quasipositive knot is concordant to infinitely many pairwise non-isotopic strongly quasipositive knots.

Sketch of proof: Let K be a sqp knot, $K \neq U$.
Let $F(\beta)$ be a Seifert surface for K that corresponds to a sqp braid β w/ $\hat{\beta} = K$.

Idea: Tie a slice knot $C \neq U$ w/ $TB(C) = -1$ (max'l Thurston-Bennequin nr.) into a positive band of $F(\beta)$.
$\rightsquigarrow$ surface F'

e.g. C = mirror of 9_{46}





Then: 1) $\partial F' = P_{\partial F(\beta)}(C)$ = satellite w/ pattern $\partial F(\beta)$ and companion C

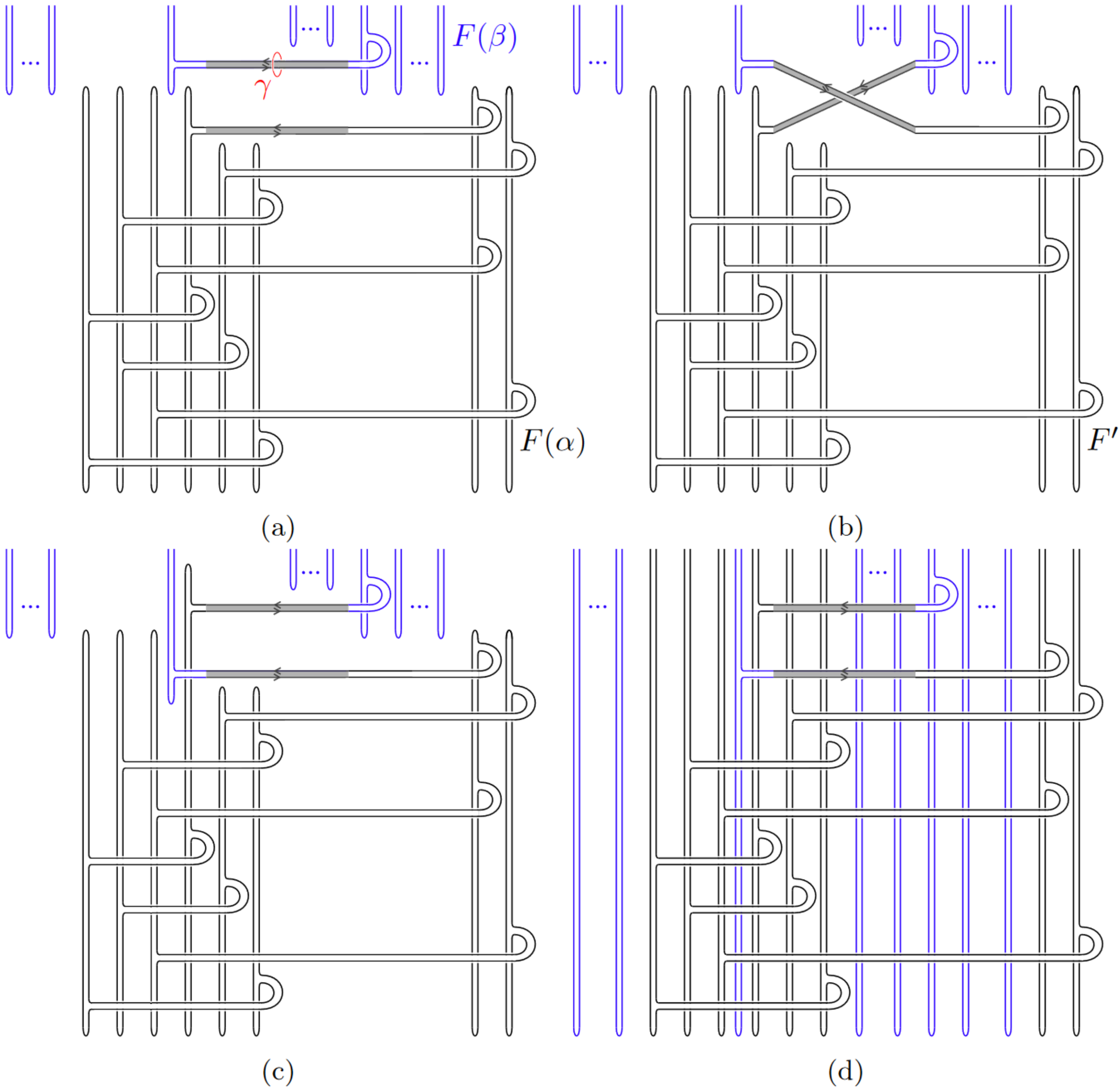
We use $C \underset{\text{conc.}}{\sim} U \Rightarrow \partial F' = P_{\partial F(\beta)}(C) \sim P_{\partial F(\beta)}(U) = \partial F(\beta) = K$

$C \underset{\text{iso}}{\not\cong} U \Rightarrow P_{\partial F(\beta)}(C) \not\cong P_{\partial F(\beta)}(U)$

Kouno–Motegi

need: geom. winding nr. of pattern ≥ 2

2) $\partial F'$ is sqp basically b/c $A(C, \underset{\substack{\| \\ TB(C)}}{-1})$ is a qp surface (Rudolph)



(a) (b) (c) (d)

We iterate this construction to obtain an infinite family of sqp knots concordant, but not isotopic to K. $\square$