Exercise Sheet 1

1. (a) Using the stereographic projection, show that S^3 is homeomorphic to the one-point compactification $\mathbb{R}^3 \cup \{\infty\}$.
   (b) The set $L = S^1 \times \{0\} \cup \{0\} \times S^1$ is a submanifold of $S^3 \subseteq \mathbb{C}^2$. Determine and visualize its image under the stereographic projection.
   (c) Let $H_i \subseteq S^3 \subseteq \mathbb{C}^2$, $i = 1, 2$, be given by $H_i = \{(z_1, z_2) \in S^3 \mid |z_i|^2 \geq \frac{1}{2}\}$. Show that $S^3 = H_1 \cup H_2$ is a decomposition of S^3 into solid tori.

Using the following Quotient Manifold Theorem, we will see in the next exercise why the lens spaces introduced in the first lecture are smooth manifolds. Let G be a group. Suppose that G is a smooth manifold such that the multiplication map $G \times G \to G, (g, h) \mapsto gh$ and the inverse map $G \to G, g \mapsto g^{-1}$ are both smooth maps. Then G is called a *Lie group*. Examples of Lie groups are the general linear groups $GL_n(\mathbb{R})$ and the orthogonal groups $O_n(\mathbb{R})$ and $SO_n(\mathbb{R})$. Any finite or countably infinite group with the discrete topology is also a Lie group.

Theorem (Quotient Manifold Theorem)**.** *Let M be a smooth manifold and let G be a Lie group with identity element e_G. Suppose that G acts on M*

(a) smoothly*, i. e. the map $G \times M \to M$ defining the group action (which fulfills $g \cdot (h \cdot p) = (gh) \cdot p$ and $e_G \cdot p = p$ for all $g, h \in G, p \in M$) is smooth,*

(b) freely*, i. e. $g \cdot p = p \Rightarrow g = e_G$, and*

(c) properly*, i. e. the map $G \times M \to M \times M, (g, p) \mapsto (g \cdot p, p)$ is a proper map. Equivalently (assuming (a) is true), for every compact subset $K \subseteq M$, the set $\{g \in G \mid (g \cdot K) \cap K \neq \emptyset\}$ is compact.*

Then the quotient space M/G is a topological manifold of dimension $\dim(M) - \dim(G)$ *and has a unique smooth structure such that the quotient map $\pi\colon M \to M/G$ is smooth (in fact, a submersion). If G is discrete and M connected, then π is a (normal) covering map.*

References: Theorem 21.10 and Chapter 21 in J. M. Lee, *Introduction to Smooth Manifolds*; see also Proposition 1.40 in A. Hatcher, *Algebraic Topology*.

2. For coprime integers p, q, consider the action of $\mathbb{Z}/p\mathbb{Z}$ on $S^3 \subseteq \mathbb{C}^2$ defined by

$$m \cdot (z, w) = \left(e^{2\pi i m/p} z, e^{2\pi i q m/p} w\right) \qquad \text{for } m \in \mathbb{Z}/p\mathbb{Z}.$$

   (a) Show that this action is smooth, free and proper. Deduce that the *lens space* $L_{p,q}$, defined as the quotient of S^3 by the above action, is a smooth 3-manifold. Note that $L_{1,0} \cong_{C^\infty} S^3$.
   (b) Use covering space theory to show that $\pi_1(L_{p,q}) \cong \mathbb{Z}/p\mathbb{Z}$.
   (c) Show that $L_{2,1} \cong_{C^\infty} \mathbb{R}\mathrm{P}^3$.

3. Let Σ be a compact surface and let $\varphi\colon \Sigma \to \Sigma$ be a diffeomorphism. Define an action of $\mathbb{Z}$ on $\Sigma \times \mathbb{R}$ via $k \cdot (x, t) = \left(\varphi^k(x), t + k\right)$.
   (a) Show that this action is smooth, free and proper. Deduce that the *mapping torus*, the quotient of $\Sigma \times \mathbb{R}$ by the above action, is a smooth 3-manifold.
   (b) Construct a 2-fold covering map $M_{\varphi^2} \to M_\varphi$ between the mapping tori.