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Exercise Sheet 2

1. (a) Find a handle decomposition for any closed orientable surface.
   (b) Draw sketches of handle cancellations of a $(k-1)$- and a k-handle for any $k \in \{1, \dots, n\}, n \in \{2, 3\}$. Indicate in your sketches also the attaching spheres, the belt spheres, the cores, the cocores and the attaching regions of the handles.

2. (a) Describe a way to compute the fundamental group of a manifold from a given handle decomposition.
   (b) The fundamental group of any compact manifold is finitely presented. Show that for any $n \geq 5$, we get any finitely presented group as the fundamental group of a closed orientable n-manifold.
   (c) Show that, on the other hand, not every finitely presented group is the fundamental group of a closed orientable 3-manifold. Groups which occur as the fundamental group of a closed orientable 3-manifolds are called *3-manifold groups*.
   Hint: Let $\langle g_1, \dots, g_n \mid r_1, \dots r_m \rangle$ be a finite presentation of a group G. We call $n-m$ the deficiency of this presentation and we define the *deficiency* of G as the maximum over the deficiencies of all finite presentations of G. Show that every 3-manifold group has non-negative deficiency and find (without proof) a group with negative deficiency.

3. Consider the 3-torus $T^3 = S^1 \times S^1 \times S^1$.
   (a) Show that we can obtain T^3 from the cube $I \times I \times I$ by identifying opposite sides.
   (b) Describe a handle decomposition of T^3 (as simple as possible).
   (c) Draw a planar Heegaard diagram of T^3.

4. Two homeomorphisms $g_0, g_1 \colon X \to X$ are *isotopic* if there exists a continuous map $F \colon X \times [0,1] \to X$ such that $f_t := F(\cdot, t)$ has the following properties: f_t is a homeomorphism for every $t \in [0,1]$, $f_0 = g_0$ and $f_1 = g_1$. Prove the following three statements.
   (a) Every homeomorphism $f \colon S^{n-1} \to S^{n-1}$ extends to a homeomorphism $F \colon D^n \to D^n$ for $n \geq 1$.
   (b) (Alexander trick) If a homeomorphism $f \colon D^n \to D^n$ restricts to the identity $\mathrm{id}_{S^{n-1}}$ on $S^{n-1} = \partial D^n$, then f is isotopic to id_{D^n}.
   Hint: Let f occur in a smaller and smaller ball and extend by the identity.
   (Probably you will also arrange that the isotopy f_t satisfies $f_t|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$ for every $t \in [0,1]$, i. e. f and id_{D^n} are isotopic *rel boundary*.)
   (c) Any manifold obtained by gluing two n-balls D^n is homeomorphic to S^n.