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Exercise Sheet 3

Exercise 2 is a bonus exercise.

1. Let p, q, r be integers such that $\gcd(p, q) = 1 = \gcd(p, r)$, $p \neq 0$. Show that there exist the following diffeomorphisms:
   (a) $L_{p,q} \cong_{C^\infty} L_{p,q+kp}$ for all $k \in \mathbb{Z}$,
   (b) $L_{p,q} \cong_{C^\infty} L_{p,r}$ if $r \equiv \pm q \pmod p$ and
   (c) $L_{p,q} \cong_{C^\infty} L_{p,r}$ if $r \equiv \pm q^{-1} \pmod p$.

2. (a) Show that the homology $H_*(M; \mathbb{Z})$ of a closed, connected, orientable 3-manifold M is determined by its fundamental group $\pi_1(M)$. *Note:* This requires some knowledge in algebraic topology. A (sketch of) proof can be found e.g. in Proposition 9.1.1 in Martelli's *An Introduction to Geometric Topology.* (∗)
   (b) Use (a) to determine the homology groups of lens spaces. Bonus: Determine the homology groups of your favorite closed, orientable 3-manifold.
   (c) Show that if M is additionally simply-connected, then $H_i(M; \mathbb{Z}) \cong H_i(S^3; \mathbb{Z})$ for all i. A closed, connected, orientable 3-manifold M with $H_i(M; \mathbb{Z}) \cong H_i(S^3; \mathbb{Z})$ for all i is called a *homology sphere.*

3. Let M be a closed, connected, orientable 3-manifold.
   (a) Find a presentation of $\pi_1(M)$ in terms of a Heegaard splitting of M.
   (b) Conclude that $\mathrm{rk}(M) \leq g(M)$, where $\mathrm{rk}(M)$ denotes the *rank* of $\pi_1(M)$, the minimal number of elements needed to generate $\pi_1(M)$.
   (c) Show that $g(T^3) = 3$.

4. Show that the Heegaard genus of $\Sigma_g \times S^1$ is $2g + 1$.