Exercise Sheet 4

Exercise 6 is a bonus exercise.

1. Let U denote the unknot, i. e. the trivial embedding $S^1 \hookrightarrow \mathbb{R}^2 \subseteq \mathbb{R}^3 \cup \{\infty\} \cong S^3$. For integers p and q, we define the (p, q)-*torus link* $T_{p,q}$ to be the link given by the curve $p\mu + q\lambda$ on the boundary of a tubular neighborhood νU of U, where μ and λ denote a meridian and Seifert longitude of U, respectively.
   (a) Draw some diagrams of torus links. Compare your results with the visualizations at `https://www.desmos.com/3d/4ba6cb90d2?lang=de` by Johannes Zander.[1]
   (b) Identify familiar links as torus links, e. g. the unknot, the trefoil knot and the Hopf links.

   Remark: Here are two facts about torus links that you do not need to prove.
   (1) A torus link is a knot if and only if p and q are coprime.
   (2) A torus knot is *trivial* (i. e. isotopic to the unknot) if and only if p or q is ± 1.

2. Let K be a knot and let νK denote a closed tubular neighbourhood of K.
   (a) Show that $S^3 \setminus \text{Int}(\nu K)$ is a homology circle, i. e. $H_*(S^3 \setminus \text{Int}(\nu K); \mathbb{Z}) \cong H_*(S^1; \mathbb{Z})$. Show that in particular $H_1(S^3 \setminus \text{Int}(\nu K); \mathbb{Z}) \cong \mathbb{Z}$ is generated by a meridian of the knot K.
   Hint: Use the Mayer-Vietoris sequence.
   Remark: If you are not familiar with (singular) homology, note that (a) also follows from (b) using the isomorphism $H_1(S^3 \setminus \text{Int}(\nu K); \mathbb{Z}) \cong \pi_1(S^3 \setminus \text{Int}(\nu K))_{\text{ab}}$ and Exercise 2 from Exercise Sheet 3.
   (b) Show that $\pi_1(S^3 \setminus \text{Int}(\nu K))$ is normally generated by a meridian of K, i. e. it is generated by the set of conjugates of a meridian.

3. A 2-component link $L = L_1 \sqcup L_2$ is called *split* if there exist two disjoint smoothly embedded balls $B_i \cong_{C^\infty} D^3 \subseteq S^3$ such that (the image of) each L_i is contained in B_i, $i = 1, 2$.
   (a) Show that if $L = L_1 \sqcup L_2$ is a split link, then $\text{lk}(L_1, L_2) = 0$.
   (b) Show (by constructing an example) that the converse of (a) is not true.

4. Show that isotopic links L_1, L_2 have homeomorphic exteriors $S^3 \setminus \text{Int}(\nu L_i)$, $i = 1, 2$.
   Hint: Use the isotopy extension theorem.

5. Let H be the positive Hopf link (which is $T_{2,2}$ in the notation from Exercise 1).
   (a) Show that $S^3 \setminus \text{Int}(\nu H)$ is diffeomorphic to $S^1 \times S^1 \times [-1, 1]$.
   (b) Show that the result of doing Dehn surgery along H with both surgery coefficients being 0 is S^3.

6. A *Seifert surface* of an oriented link L is an oriented, connected, compact surface smoothly embedded in S^3 with oriented boundary L. Describe an algorithm for constructing a Seifert surface of an oriented link from one of its diagrams. (∗)

   Hint: First resolve the crossings appropriately and fill the remaining circles with disks. Then try to glue the disks by drilled bands to obtain a Seifert surface of the original link.

   Second hint/solution: This algorithm, called *Seifert's algorithm*, can be found in almost any introductory book on knot theory, see for example Theorem 5.A.4 in D. Rolfsen, *Knots and Links*, or Theorem 4.3.4 in J. Schultens, *Introduction to 3-Manifolds*.

[1] This link and many nice link diagrams can be found in the lecture notes on Knot Theory by Stefan Friedl available from his website (see "topology notes").