**Concordance of positive braid knots**

Paula Truöl

This talk is concerned with the following question.

**Question 1.** *Are concordant positive braid knots isotopic?*

We study *knots* in the 3-sphere $S^3$, i.e. non-empty, connected, oriented, closed, smooth 1-dimensional submanifolds of $S^3$, considered up to ambient isotopy. Two knots $K$ and $J$ are called *concordant* if there exists an annulus $A \cong S^1 \times [0,1]$ smoothly and properly embedded in $S^3 \times [0,1]$ such that $\partial A = K \times \{0\} \cup J \times \{1\}$ and such that the induced orientation on the boundary of the annulus agrees with the orientation of $K$, but is the opposite one on $J$. Knots up to concordance form a group, the *concordance group* $\mathcal{C}$, with the group operation induced by connected sum. A knot is concordant to the unknot if and only if it is *slice*, i.e. if it bounds a smoothly embedded 2-dimensional disk $D^2$ in $B^4$, the 4-ball bounded by $S^3$.

Clearly, isotopic knots are concordant. The converse is in general not true as any nontrivial slice knot shows. For example, for any nontrivial knot $K$ the knot $K \# -K$ is slice. Here $-K$ denotes the *inverse* of $K$ in $\mathcal{C}$, the image of $K$ under an orientation-reversing diffeomorphism of $S^3$ with the opposite orientation.

However, it was shown by Litherland [6] that *algebraic knots*, which are knots of isolated singularities of complex algebraic plane curves, are isotopic if they are concordant. This naturally leads to Question 1 when looking at the following set of inclusions. We have

$$\{\text{positive torus knots}\} \subset \{\text{algebraic knots}\} \subset \{\text{positive braid knots}\}.$$

Note that the torus knots $T_{p,q}$ for coprime positive integers $p$ and $q$ are obtained as knots associated to the singularity $z^p - w^q = 0$ for $z, w \in \mathbb{C}$. Algebraic knots are certain iterated cables of torus knots and they are known to be *positive braid knots*, i.e. they can be obtained as closures of positive braids.

By a fundamental theorem of Alexander [1], every knot in $S^3$ can be represented as the closure of an $n$-braid for some positive integer $n$. Here, an $n$-*braid* is an element of the *braid group on $n$ strands*, denoted $B_n$, whose classical presentation with $n-1$ generators $\sigma_1, \dots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2, \text{ and } \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

was introduced by Artin [2]. A *positive braid* is an element of the braid group $B_n$ for some $n$ that can be written as a positive braid word $\sigma_{s_1} \sigma_{s_2} \cdots \sigma_{s_l}$ with $s_i \in \{1, \dots, n-1\}$. The set of positive braid knots is a subset of the set of *positive knots*, i.e. the knots that admit a diagram with only positive crossings.

Let $\mathrm{wr}(\gamma)$ denote the *writhe* of a braid $\gamma \in B_n$ for some $n > 0$, i.e. the exponent sum of the word $\gamma$ in the generators $\sigma_1, \dots, \sigma_{n-1}$. If $\gamma$ is a positive braid such that its closure $K = \widehat{\gamma}$ is a knot, then, by work of Bennequin [4] and Rudolph [7] — the latter building on Kronheimer and Mrowka's proof of the local Thom conjecture

[5] — we have

$$g_4(K) = g(K) = \frac{\mathrm{wr}(\gamma) - n + 1}{2}. \tag{1}$$

Here $g(K)$ denotes the 3-*genus of* $K$, the minimal genus of a compact, connected, oriented smooth surface in $S^3$ with oriented boundary the knot $K$, and $g_4(K)$ denotes the 4-*genus* of $K$, the minimal genus of a compact, connected, oriented surface smoothly embedded in $B^4$ with oriented boundary the knot $K$ in $S^3 = \partial B^4$. A corollary of Equation (1) is that there can be only finitely many positive braid knots in each concordance class in $\mathcal{C}$. In fact, by a result of Baader, Dehornoy and Liechti [3] this is true in more generality: every concordance class in $\mathcal{C}$ contains at most finitely many isotopy classes of positive knots. For positive braid knots, this follows by combining Equation (1) with the facts that the 4-genus is a concordance invariant for positive braid knots and that the writhe of a positive braid $\gamma$ equals the number of generators in the corresponding braid word and is linearly bounded from below by twice the *positive braid index* of $\widehat{\gamma}$ — the minimal number of strands among the positive braid representatives of $\widehat{\gamma}$. The question whether there is indeed only one isotopy class of positive braid knots of fixed braid index in each concordance class remains open. We are particularly interested in this question when the braid index is fixed to be 3, the first interesting case.

## References


[1] J. W. Alexander. *A lemma on systems of knotted curves*, Proceedings of the National Academy of Sciences of the United States of America **9(3)** (1923), 93–95.

[2] E. Artin, *Theorie der Zöpfe*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **4** (1925), 47–72.

[3] S. Baader, P. Dehornoy, and L. Liechti, *Signature and concordance of positive knots*, Bulletin of the London Mathematical Society **50** (2017), 166–173.

[4] D. Bennequin, *Entrelacements et équations de Pfaff, in Third Schnepfenried geometry conference*, Vol. 1 (Schnepfenried, 1982), vol. 107 of Astérisque, Soc. Math. France, Paris, 1983, 87–161.

[5] P. B. Kronheimer and T. S. Mrowka, Gauge theory for embedded surfaces. I, Topology **32** (1993), 773–826.

[6] R. A. Litherland, *Signatures of iterated torus knots*, Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977), Lecture Notes in Math **722**, 17–84. Springer, Berlin, 1979.

[7] L. Rudolph, *Quasipositivity as an obstruction to sliceness*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), 51–59.