NOTES OF 16 MAI 2018 : EQUIVARIANT TWISTED $\mathcal{D}$-MODULES AND ADMISSIBLE ORBITS

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1. general setting

Let G be an abelian group, H a torus with an action $\kappa : G \to \mathrm{Aut}(H)$, X be a smooth G-variety and $\widetilde{X} \to X$ be a G-equivariant H-torsor.

We assume that the action of G on X has only a finite number of orbits. We write $X = \bigsqcup_{i \in I} Q_i$, the decomposition into orbits. For $i_1, i_2 \in I$, we denote $i_1 \leq i_2$ if $Q_{i_1} \subseteq \overline{Q}_{i_2}$. We say a subset $J \subseteq I$ is **closed** if $j \in J$ and $j' \leq j$ implies $j' \in J$. For each $i \in I$, we denote $\bar{i} = \{j \in I \ ; \ j \leq i\} \subseteq I$ the **closure** of i.

Recall that we have the classifiation of simple G-equivariant monodromic $\mathcal{D}$-modules on X.

2. flag variety

We take X as flag variety G/B, on which $N = \mathrm{Rad}(B)$ acts by left multiplication, and $\widetilde{X} = G/N$ being N-equivariant $H = B/N$-torsor over X. We shall study the N-equivariant H-monodromic $\mathcal{D}$-modules on X. Let W be the Weyl group.

We fix an anti-dominant regular weight $\lambda \in \mathfrak{h}^*$. The Beĭlinson–Bernstein localisation theorem implies that

Théorème 2.1. — $\mathrm{Mod}\left(\mathrm{U}(\mathfrak{g}), N\right)_\lambda \cong \mathrm{Mod}\left(\widetilde{\mathcal{D}}_X, N\right)_\lambda$ *is W-stratified.*

3. Admissible orbits

Let K be an algebraic group with a morpism $K \to G$. Let $x \in X$ be a point and let $K_x = \mathrm{Stab}_K(x)$. Choose any $\widetilde{x} \in \widetilde{X}$ which lifts x. Then $K^0_{\widetilde{x}} \subseteq K_x$ is a normal subgroup and that $K_{(x)} = K_x / K^0_{\widetilde{x}}$ doesn't depends on the choice of $\widetilde{x}$. The Lie algebra $\mathfrak{k}_{(x)}$ is naturally embedded into $\mathfrak{h}$.

Définition 3.1. — For $x \in X$, we define

$$\mathfrak{h}^*(x) = \left\{\varphi \in \mathfrak{h}^* \ ; \ \kappa(K_(x))\varphi = \varphi, \ \varphi\,|_{\mathfrak{k}_{(x)}} = 0\right\}$$

The subset $\mathfrak{h}^*(x)$ depends only on the orbit $Q = Kx$. We shall denote $\mathfrak{h}^*(Q) = \mathfrak{h}^*(x)$. We define $\mathfrak{h}^*_{\mathbf{Z}}(Q) = \mathfrak{h}^*(Q) \cap \mathbf{X}^*(H)$.

Lemme 3.2. — *For $\varphi \in \mathfrak{h}^*_{\mathbf{Z}}$. Then $\varphi \in \varphi \in \mathfrak{h}^*_{\mathbf{Z}}(Q)$ if and only if there exits a K-invariant regular function $f_\varphi \neq 0$ on $\widetilde{Q} = \pi^{-1}(Q)$ such that for all $\widetilde{x} \in \widetilde{Q}$,*

$$f_\varphi\left(h\widetilde{x}\right) = \left(\exp\varphi\right)(h) f_\varphi(\widetilde{x}).$$

Définition 3.3. — The function f_φ is called $\widetilde{Q}$**-positive** if $f_\varphi \in \mathcal{O}\left(\overline{\widetilde{Q}}\right)$ and $f_\varphi^{-1}(0) = \overline{\widetilde{Q}} \smallsetminus \widetilde{Q}$ (i.e. f_φ extends holomorphically by zero to the closure.)

Définition 3.4. — The orbit Q is called **admissible** if there exits $\varphi \in \mathfrak{h}^*_{\mathbf{Z}}(Q)$ such that f_φ is $\overline{Q}$-positive. We denote $\mathfrak{h}^{*+}_{\mathbf{Z}}(Q)$ the subset of $\mathfrak{h}^*_{\mathbf{Z}}(Q)$ consisting of $\overline{Q}$-positive weights.

Now consider the action of N on $X = G/B$ by left multiplication. Then $X = \bigsqcup_{w \in W} Q_w$ with $Q_w = NwB$ being the Schubert cells.

Proposition 3.5. — *For all $w \in W$, the orbit Q_w is admissible.*

Démonstration. — Since $\rho + \mathfrak{h}^{*+} \subseteq \mathfrak{h}^{*+}(Q_w)$, the latter is nonempty.

Take any $\chi \in \rho + \mathfrak{h}^{*+}$, the irreducible G-module $L(\chi)$ has a lowest weight vector $v \in V$. Consider the morphism

$$G \to V$$
$$g \mapsto gv,$$

which induces $q_v : \widetilde{X} = G/N \to V$. Let $\dot{w} \in N_G(T)$ be a lifting of w. We choose a linear functional $l \in V^*$ such that $l(\dot{w}v) \neq 0$ and that $l\left(n\dot{w}v\right) = 0$ for any $n \in \operatorname{Lie} N$.

Then we put $f = \ell \circ q_v$, so that $f\left(\overline{\widetilde{Q}} \smallsetminus \widetilde{Q}\right) = 0$ and $f\left(\widetilde{Q}\right) \neq 0$. □

4. nearby cycles

We have the following diagram of two cartesian squares

$$\begin{array}{ccccc} Z & \longrightarrow & Y & \longleftarrow & U \\ \big\downarrow f & & \big\downarrow f & & \big\downarrow f \\ \{0\} & \longrightarrow & \mathbf{A}^1 & \longleftarrow & \mathbf{A}^1 \smallsetminus \{0\} \end{array} .$$

Recall that we have defined for each $a \in \mathbf{N}$ a functor $\pi^a_f : \operatorname{Mod}(\mathcal{D}_U) \to \operatorname{Mod}(\mathcal{D}_Y)$, that we denoted $\pi^0_f = \Psi^{\text{uni}}_f$, $\pi^1_f = \Xi_f$ and there are exact sequences

$$0 \to j_! \to \Xi_f \overset{s}{\twoheadrightarrow} \Psi^{\text{uni}}_f \to 0$$
$$0 \to \Psi^{\text{uni}}_f \to \Xi_f \to j_* \to 0$$

We consider the composite $s : \Xi_f \to \Psi^{\text{uni}}_f \to \Xi_f$.

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