NOTES OF 28 MARCH 2018 : $\mathcal{D}$-MODULES II

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1. Setting

Let X be a smooth variety over $\mathbf{C}$, $\mathcal{O}_X$ the structure sheaf, Θ_X the tangent bundle and $\mathcal{D}_X \subseteq \mathcal{E}\mathrm{nd}_{\mathbf{C}}(\mathcal{O}_X)$ the sheaf of differential operators.

2. Recall of non-derived pull-back and push-forward

Let $f : X \to Y$ be a morphism of smooth varieties. To f we can associate two functors : pull-back

$$M_l(\mathcal{D}_X) \ni f^*M \leftarrow M \in M_l(\mathcal{D}_Y)$$

Définition 2.1. — We put $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$ and $f^*M = \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}M$. This is the pull-back of left $\mathcal{D}$-modules.

and push-forward

$$M_r(\mathcal{D}_X) \ni N \mapsto f_*N \in M_r(\mathcal{D}_Y)$$

Définition 2.2. — We put $f^*N = {}^{\mathcal{O}}f_*(N \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$. This is the push-forward of right $\mathcal{D}$-modules.

For example, if $j : U \hookrightarrow X$ is an open embedding, then j^* is the restriction to the open subset U, whereas j_* turns $\mathcal{D}_U$-modules into $\mathcal{D}_X$-modules via the localisation of ring $\mathcal{D}_X \to \mathcal{D}_U$.

For closed embedding $i : X \hookrightarrow Y$ such that $X = \{y_{r+1} = \cdots = y_n = 0\}$, we have

$$i_*M = \bigoplus_{i_1,\ldots,i_{n-r}\geq 0} M\partial_{r+1}^{i_1}\ldots\partial_n^{i_{n-r}}$$

3. Derived pull-back and push-forward

Définition 3.1. — The derived pull-back is

$$f^* : \mathrm{D}^b(M_l(\mathcal{D}_Y)) \to \mathrm{D}^b(M_l(\mathcal{D}_X))$$
$$M \mapsto \mathcal{D}_{X\to Y} \otimes^{\mathrm{L}} M$$

Définition 3.2. — The derived push-forward is

$$f_* : \mathrm{D}^b(M_r(\mathcal{D}_X)) \to \mathrm{D}^b(M_r(\mathcal{D}_Y))$$
$$M \mapsto \mathrm{R}^{\mathcal{O}} f_*\left(M \otimes^{\mathrm{L}} \mathcal{D}_{X\to Y}\right)$$

Here the index $\mathcal{O}$ is to distinguish the usual push-forward of coherent $\mathcal{O}_X$-modules from the push-forward of $\mathcal{D}_X$-modules.

We have composition laws $(f\circ g)^* = g^* \circ f^*$ and $(f\circ g)_* = f^* \circ g^*$.

Remarque 3.3. — Whenever f is smooth, f^* is exact.

4. De Rham and Spencer complex

Recall that we have the de Rham complex $\mathrm{dR}(\mathcal{D}_X)$:

$$0 \to \Omega_X^0 \otimes \mathcal{D}_X \xrightarrow{d} \Omega_X^1 \otimes \mathcal{D}_X \to \ldots \to \Omega_X^{d_X} \otimes \mathcal{D}_X \to 0$$

Proposition 4.1. — $\mathrm{dR}(\mathcal{D}_X)$ *is a resolution of* $\Omega_X^{d_X}$ *in* $\mathrm{D}^b(\mathrm{Mod}_r(\mathcal{D}_X))$.

Corollaire 4.2. — $\mathrm{dR}(M) \cong \Omega_X^{d_X} \otimes^{\mathrm{L}} M$ *in* $\mathrm{D}^b(\mathcal{D}_X)$.

Dually, we have the Spencer complex $\mathrm{Sp}(\mathcal{D}_X)$:

$$0 \to \mathcal{D}_X \otimes \bigwedge^{d_X} \Theta_X \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{D}_X \otimes \bigwedge^{0} \Theta_X \to 0$$

Proposition 4.3. — $\mathrm{Sp}(\mathcal{D}_X)$ *is a resolution of* $\mathcal{O}_X$ *in* $\mathrm{D}^b(\mathrm{Mod}_l(\mathcal{D}_X))$.

5. Coherent $\mathcal{D}$-modules

Recall that $\mathcal{D}_X$ is equipped with a filtration by order

$$\mathcal{D}_X^0 = \mathcal{O}_X$$
$$\mathcal{D}_X^1 = \mathcal{O}_X \oplus \Theta_X$$
$$\mathcal{D}_X^i \mathcal{D}_X^j \subseteq \mathcal{D}_X^{i+j}$$
$$\left[\mathcal{D}_X^i, \mathcal{D}_X^j\right] \subseteq \mathcal{D}_X^{i+j-1}$$

We can introduce filtrations on modules :

Définition 5.1. — A filtration on a $\mathcal{D}_X$-module M is a sequence of abelian subsheaves of M

$$\ldots \subseteq F_i M \subseteq F_{i+1} M \subseteq \ldots$$

such that $F_i M = 0$ for $i \ll 0$, that $M = \bigcup_i F_i M$, and that $\mathcal{D}_X^i F_j M \subseteq F_{i+j} M$.

For any filtred module M, we put

$$\mathrm{gr}^F M := \bigoplus_i F_i M / F_{i-1} M$$

Théorème 5.2. — *We have a canonical isomorphism* $\mathrm{gr}\,\mathcal{D}_X \cong {}^{\mathcal{O}}\pi_* \mathcal{O}_{T^*X}$.

Sketch of proof. — Take any coordinate chart $x_1, \cdots, x_n$, so that we have $\xi_i \in \mathcal{O}_{T^*X}$ and $\partial_i \in \mathcal{D}_X$.

We introduce the **principal symbols**. Let $P \in F_l \mathcal{D}_X$, $P = \sum_{|\alpha|\le l} a_\alpha \partial^\alpha$. Then $P \mod F_{l-1} = \sum_{|\alpha|=l} a_\alpha \xi^\alpha$.

We have an assignment $\mathcal{D}_X \ni P \mapsto \overline{P} \in \mathrm{gr}\,\mathcal{D}_X \cong \mathcal{O}_X[\xi_1, \cdots, \xi_n] = \mathrm{Sym}(\Theta_X)$. □

Définition 5.3. — Let M be a filtred $\mathcal{D}_X$-module with a filtration F. We say that F is **good** if $\operatorname{gr}^F M$ is a coherent $\operatorname{gr}\mathcal{D}_X$-module.

Théorème 5.4. — *For a quasi-coherent $\mathcal{D}_X$-module M, we have*

- *If M admits a good filtration, then it is coherent $\mathcal{D}_X$-module*
- *Conversely*

6. Characteristic variety and singular support

Let M be a coherent $\mathcal{D}_X$-module. Let F be a good filtration on M. We put $\widetilde{\operatorname{gr} M} = \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\operatorname{gr}\mathcal{D}_X} \pi^{-1}\operatorname{gr}^F M$, which is an $\mathcal{O}_{T^*X}$-module.

Définition 6.1. — The support of $\operatorname{gr}^F M$, defined by the radical of its annihilator in $\mathcal{O}_{T^*X}$, is called the **characteristic variety** of M, denoted $\operatorname{char}(M) \subseteq T^*X$.

Proposition 6.2. — *Let $0 \to M \to N \to L \to 0$ be an exact sequence of left $\mathcal{D}_X$-modules. Then* $\operatorname{char}(N) = \operatorname{char}(M) \cup \operatorname{char}(L)$.

Théorème 6.3. — *Let $f : X \to Y$. A sufficient condition on f for that f^* preverses the coherence of a $\mathcal{D}_Y$-module M is that f is* **non-characterstic** *for M.*

Théorème 6.4. — *When f is proper, f_* preserves the coherence.*

7. Kashiwara Theorem

Théorème 7.1 **(Kashiwara).** — *For a closed embedding $i : Z \hookrightarrow X$, the direct image* $i_* : \operatorname{Mod}(\mathcal{D}_Z) \to \operatorname{Mod}^Z(\mathcal{D}_X)$ *is an equivalence of categories, where* $\operatorname{Mod}^Z(\mathcal{D}_X)$ *is the category of $\mathcal{D}_X$-modules which is supported on Z when viewed as $\mathcal{O}_X$-module.*

8. Holonomicity

In order to remedy to the problem of coherence not being preserved in general, we introduce a smaller class of $\mathcal{D}$-modules.

Proposition 8.1. — *For each irreducible component of* $\operatorname{char}(M) \subseteq T^*X$ *is of dimension* $\geq \dim X$.

Démonstration. — Assume $\dim\operatorname{char}(M) < \dim X$. Then M is supported on some $i : Z \hookrightarrow X$ of dimension $< \dim X$. Then by the theorem of Kashiwara, there exists $N \in \operatorname{Mod}(\mathcal{D}_Z)$ such that $M \cong i_*N$. Let $\delta_N = \dim\operatorname{char}(N) - \dim Z$. This number is preserved under direct image. Then we have

$$\delta_M = \delta_N < 0$$

contradiction. □

Définition 8.2. — A $\mathcal{D}_X$-module is called **holonomic** if $\dim\operatorname{char}(M) = \dim X$.

Proposition 8.3. — *The holonomicity is preserved by extensions, submodules, quotients, direct images and inverse images.*

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