NOTES OF 2 MAI 2018 : HARISH-CHANDRA MODULES, MONODROMIC $\mathcal{D}$-MODULES

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1. Goal

The goals of the talk today is to

(i). explain the Harish-Chandra algebras and Harish-Chandra modules, weak and strict
(ii). explain the monodromic construction of twisted differential algebras

We follow Beĭlinson–Bernstein.

2. Harish-Chandra modules

Let G be an algebraic group over $\mathbf{C}$ and X be a variety equipped with a G-action $\mu : G \times X \to X$.

2.1. infinitesimal equivariance. — Denote $\mathcal{T}_X \subseteq \mathcal{E}\mathrm{nd}_{\mathbf{C}}(\mathcal{O}_X)$ the tangent sheaf of X. Let $\mathfrak{g} = \mathrm{Lie}\, G$. There is an infinitesimal action $\alpha : \mathfrak{g} \to \mathcal{T}_X$. We form the Lie algebroid $\widetilde{\mathfrak{g}}_X = \mathcal{O}_X \otimes_{\mathbf{C}} \mathfrak{g}$ by setting

$$\begin{aligned} \sigma : \widetilde{\mathfrak{g}}_X &\to \mathcal{T}_X \\ f \otimes \gamma &\mapsto f\alpha(\gamma) \\ [-,-] : \widetilde{\mathfrak{g}}_X \times \widetilde{\mathfrak{g}}_X &\to \widetilde{\mathfrak{g}}_X \\ (f_1 \otimes \gamma_1, f_2 \otimes \gamma_2) &\mapsto f_1 f_2 \otimes [\gamma_1, \gamma_2] \\ &\qquad + f_1 \alpha(\gamma_1)(f_2)\gamma_2 - f_2\alpha(\gamma_2)(f_1)\gamma_1 \end{aligned}$$

Let $P \in \mathrm{QCoh}^G(X)$ be a G-equivariant quasi-coherent $\mathcal{O}_X$-module. The Lie algebroid of infinitesimal symmetries of P is a sub-sheaf $\widetilde{\mathcal{T}}_P \subseteq \mathcal{T}_X \times \mathcal{E}\mathrm{nd}_{\mathbf{C}}(P)$ defined by

$$\widetilde{\mathcal{T}}_P = \{(\xi, \varphi) \in \mathcal{T}_X \times \mathcal{E}\mathrm{nd}_{\mathbf{C}}(P) \ ; \ \varphi(fs) = \xi(f)s + f\varphi(s)\}.$$

together with the first projection $\sigma = \mathrm{pr}_1 : \widetilde{\mathcal{T}}_P \to \mathcal{T}_X$. The G-equivariant structure on P induces a morphism of Lie algebroids $\alpha_P : \widetilde{\mathfrak{g}}_X \to \widetilde{\mathcal{T}}_P$.

Let M be a quasi-coherent $\mathcal{O}_X$-module.

Définition 2.2. — A $\mathfrak{g}$-action on M is a morphism of Lie algebroids $\widetilde{\mathfrak{g}}_X \to \widetilde{\mathcal{T}}_M$, or equivalently, it is a morphism of Lie algebras $\alpha_M : \mathfrak{g} \to \mathrm{End}_{\mathbf{C}}(M)$ such that

$$\alpha_P(\gamma)(fm) = f\alpha_P(\gamma)(m) + \alpha(\gamma)(f)m.$$

We denote $\mathrm{QCoh}^{\mathfrak{g}}(X)$ the category of quasi-coherent $\mathcal{O}_X$-module equipped with a $\mathfrak{g}$-action.

2.3. G-equivariant differential bi-modules. — Let M be an $\mathcal{O}_X$-differential bi-module on X i.e. a quasi-coherent $\mathcal{O}_{X\times X}$-module supported on the diagonal.

Let $G^\wedge \hookrightarrow G \times G$ be the formal completion of $G \times G$ along the diagonal G_Δ, which is a group formal scheme acting on $X \times X$.

Définition 2.4. — A G**-action** on M as differential bi-module is an action of $G^\wedge$ on the M.

Since in characteristic 0, formal groups are determined by their Lie algebra, one can describe a G-action on M in terms of Lie algebras.

Proposition 2.5. — *A G-action on M is equivalent to a pair (μ_M, α_M), where μ_M is a G-equivariant structure on M, with G acting on $X \times X$ diagonally, and α_M is a $(\mathfrak{g} \times \mathfrak{g})$-action on M, such that the differential of μ_M is agrees with α_M restricted to the diagonal $\mathfrak{g} \subseteq \mathfrak{g} \times \mathfrak{g}$.*

In summary, the category of G-equivariant differential bimodules on X is the 2-fibred product

$$\mathrm{QCoh}^{G_\Delta}_{X_\Delta}(X \times X) \times_{\mathrm{QCoh}^{\mathfrak{g}_\Delta}_{X_\Delta}(X\times X)} \mathrm{QCoh}^{\mathfrak{g}\times\mathfrak{g}}_{X_\Delta}(X \times X)$$

which is equivalent to $\mathrm{QCoh}^{G^\wedge}_{X_\Delta}(X \times X)$.

2.6. Harish-Chandra algebras. — Let $\mathcal{A}$ be a differential algebra, i.e. differential bi-module $\mathcal{A} \in \mathrm{QCoh}_{X_\Delta}(X \times X)$ with a multiplicative structure together with a morphism of algebras $\mathcal{O}_{X_\Delta} \to \mathcal{A}$ which is compatible with $\mathcal{O}_{X\times X}$-module structure.

Définition 2.7. — A G**-action** on $\mathcal{A}$ is a G-action $(\mu_{\mathcal{A}}, \alpha_{\mathcal{A}})$ as $\mathcal{O}_X$-differential bimodule such that

(i). $(\mathcal{A}, \mu_{\mathcal{A}}) \in \mathrm{Alg}^G(X \times X)$ and
(ii). $\alpha_{\mathcal{A}}(\gamma, 0)(a_1a_2) = \alpha_{\mathcal{A}}(\gamma, 0)(a_1)a_2$ and $\alpha_{\mathcal{A}}(0, \gamma)(a_1a_2) = a_1\alpha_{\mathcal{A}}(\gamma, 0)(a_2)$.

A differential algebra with a G action is called a **Harish-Chandra algebra** or a $(\mathcal{O}_X, G)$**-differential algebra**.

Remarque 2.8. — One can similarly define Harish-Chandra Lie algebebroid in such a way that there is an adjunction

$$\{\text{HC-Lie algebroid}\} \underset{\mathrm{Lie}}{\overset{\mathrm{U}}{\rightleftarrows}} \{\text{HC-algebra}\}$$

which enhances the adjunction Lie-algebroids–differential-algebras.

2.9. Harish-Chandra modules. — Let $(\mathcal{A}, \mu_{\mathcal{A}}, \alpha_{\mathcal{A}})$ be a Harish-Chandra algebra.

Définition 2.10. — (i). The category of **weak $(\mathcal{A}, G)$-modules** is $\mathrm{Mod}(\mathcal{A}, G)_{\mathrm{weak}} = \mathrm{QCoh}^G(\mathcal{A})$.

(ii). An **$(\mathcal{A}, G)$-module** is a weak $(\mathcal{A}, G)$-module M such that for all $\gamma \in \mathfrak{g}$, the action of $\alpha_{\mathcal{A}}(\gamma, 0)(1) \in \mathcal{A}$ agrees with the infinitesimal equivariance $\alpha_M(\gamma)$ induced by the G-equivariance of M. The category of $(\mathcal{A}, G)$-modules is denoted $\mathrm{Mod}(\mathcal{A}, G)$

Exemple 2.11. — When $X = \mathrm{pt}$ and $\mathcal{A} = \mathbf{C}_X$, the weak and strict $(\mathcal{A}, G)$-modules are $\mathrm{Mod}(\mathcal{A}, G)_{\mathrm{weak}} = \mathrm{Rep}(G)$ and $\mathrm{Mod}(\mathcal{A}, G) \cong \mathrm{Rep}(\pi_0(G))$.

The tensor product $\mathcal{A} \otimes_{\mathbf{C}} \mathrm{U}(\mathfrak{g})$ is a Harish-Chandra algebra in a natural way.

Lemme 2.12. — *Weak $(\mathcal{A}, G)$-modules are the same as $(\mathcal{A} \otimes_{\mathbf{C}} \mathrm{U}(\mathfrak{g}), G)$-modules.*

Lemme 2.13. — *(i). Harish-Chandra algebras on X are the same as differential algebras on $[G\backslash X]$.*

(ii). Let $\mathcal{A}$ be a Harish-Chandra algebra. An $(\mathcal{A}, G)$-module is the same as a module over the algebra on the stack $[G\backslash X]$.

In the case where $\pi : X \to Z$ is a G-torsor and $\mathcal{A}$ is a Harish-Chandra algebra on X, we define $\widetilde{\mathcal{A}}_Z = (\pi_* \mathcal{A})^G$. The corresponding differential algebra on the quotient $[G\backslash X] = Z$ is $\mathcal{A}_Z = \widetilde{\mathcal{A}}_Z / \widetilde{\mathfrak{g}}_Z \cdot \widetilde{\mathcal{A}}_Z$ where $\widetilde{\mathfrak{g}}_Z = (\pi_* \mathcal{O}_X \otimes \mathfrak{g})^G$ is the "vertical part".

Given any differential algebra $\mathcal{A}_Z$ on Z, defining $\mathcal{A} = \pi^* \mathcal{A}_Z$, the equivalence on weak modules is given by

$$\begin{aligned} \mathrm{Mod}(\mathcal{A}, G)_{\mathrm{weak}} &\xrightarrow{\cong} \mathrm{Mod}\left(\widetilde{\mathcal{A}}_Z\right) \\ M &\longmapsto (\pi_* M)^G \\ \pi^{-1}\left(\pi_* \mathcal{A} \otimes_{\widetilde{\mathcal{A}}_Z} N\right) &\longleftarrow N \end{aligned}$$

which induces an equivalence of subcategories

$$\mathrm{Mod}(\mathcal{A}, G) \xrightarrow{\cong} \mathrm{Mod}(\mathcal{A}_Z)$$

3. Monodromic twisted $\mathcal{D}$-modules

3.1. comparison with untwisted $\mathcal{D}$-modules. — Let H be a torus X be a smooth variety and $\pi : \widetilde{X} \to X$ be an H-torsor. We have $\widetilde{D}_X = \left(D_{\widetilde{X}}\right)^H$. From the previous discussion, there are equivalences $\mathrm{Mod}\left(\widetilde{\mathcal{D}}_X\right) \cong \mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}}, H\right)_{\mathrm{weak}}$ and $\mathrm{Mod}(\mathcal{D}_X) \cong \mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}}, H\right)$.

Let $\mathfrak{h}_{\mathbf{Z}} = \mathrm{Hom}(H, \mathbf{C}^\times) \subseteq \mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*$, we denote $\overline{\lambda} = \lambda + \mathfrak{h}^*_{\mathbf{Z}} \subseteq \mathfrak{h}$.

The equivalence

$$\mathrm{Mod}\left(\left(\pi_* \mathcal{D}_{\widetilde{X}}\right)^H\right) \cong \mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}}, H\right)_{\mathrm{weak}}$$

sends $\mathrm{Mod}_{\mathrm{fin}}\left(\left(\pi_* \mathcal{D}_{\widetilde{X}}\right)^H\right)$ to a subcategory $\mathrm{Mod}_{\mathrm{fin}}\left(\mathcal{D}_{\widetilde{X}}, H\right)_{\mathrm{weak}}$.

There is a forgetful functor $o : \mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}}, H\right)_{\mathrm{weak}} \to \mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}}\right)$ which sends $\mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}}^{\lambda}\right)$ to $\mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}}\right)_{\overline{\lambda}}$.

Lemme 3.2. — *The functor*

$$o : \mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}}^{\lambda}\right) \to \mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}}\right)_{\overline{\lambda}}$$

is an equivalence.

Remarque 3.3. — We have thus a characterisation of $\mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}}^{\lambda}\right)$ as a full subcategory of the category of $\mathcal{D}$-modules on $\widetilde{X}$, which allows to transmit the standard results of $\mathcal{D}$-modules to the category $\mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}}^{\lambda}\right)$.

There is also a version for $\widetilde{\lambda}$.

3.4. equivariant monodromic $\mathcal{D}$-modules. — Let G be another algebraic group with $\pi_0(G)$ acting on H. Let $\widetilde{X}$ be a $G \ltimes H$-variety such that $\pi : \widetilde{X} \to X$ is an H-torsor. Let $\mathrm{Mod}\left(\widetilde{\mathcal{D}}_X, G\right)$ be the category of weak $\left(\mathcal{D}_{\widetilde{X}}, G \ltimes H\right)$-modules which are strong along G.

Lemme 3.5. — *For $\lambda \in \mathfrak{h}^*$, let $G_\lambda = \mathrm{Stab}_G(\lambda)$ and let $\lambda^{\cdot}$ be its G-orbit. There is an equivalence*

$$\mathrm{Mod}\left(\mathcal{D}_{\widetilde{X}}, G\right)_{\widetilde{\lambda^{\cdot}}} \cong \mathrm{Mod}\left(\mathcal{D}_X, G_\lambda\right)_{\widetilde{\lambda}}$$

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