Synthetic spectra II

Last week, Yuqing talked about:

- Grothendieck topologies,
- Additive ∞-sites,
- Spherical sheaves,
- t-structures on $\mathrm{Shv}^{Sp}_{\Sigma}(\mathcal{C})$
- monoidal structure on —"—.

This week: I'll talk about:

1. Motivation from abelian categories,
2. Adams type homology theories,
   ↳ comodules, Hopf algebroids
3. $\boxed{\text{SYNTHETIC SPECTRA}}$
   ↳ The functor $\nu : Sp \longrightarrow Syn_E$
   ↳ The bigraded homotopy groups
4. Hypercomplete sheaves in Syn_E

1. Sheaves on abelian categories

Recall a <u>Grothendieck abelian category</u> is a cocomplete abelian category A generated under colimits by a set of compact generators and where filtered colimits are exact.

<u>Def^{n}</u>: A choice of compact generators is an additive subcategory $P \subseteq A$ s.t. $0 \in P$

(i) objects of P are compact;
(ii) P generates A by colimits; and
(iii) closed under pb along epis.

Given such a $P \subseteq A$, we say $\{Q_i \to P\}$ inside P is a cover for the epimorphism (pre)topology if $|\{Q_i \to P\}| = 1$ ☆ and $Q \to P$ is an epi.

<u>Notes</u>:

- P is an add. site.
- ☆ is useful

Defn: Given $\Phi \subseteq \mathcal{A}$, then we call $\mathcal{P}^{Set}_{\Sigma}(\Phi)$ its projective envelope.

Prop: [2.53] Given the above, $\mathcal{P}^{set}_{\Sigma}(\Phi)$ is Grothendieck abelian, generated by the compact, projective objects $y(P)$ under

$$y : P \hookrightarrow \mathcal{P}^{set}_{\Sigma}(\Phi).$$

(This justifies the name!)

Pf: ∘ P is add $\rightsquigarrow$ $\mathcal{P}^{Set}_{\Sigma}(P) \simeq \mathcal{P}^{Ab}_{\Sigma}(P)$.

$$\text{abelian subcat.} \longrightarrow \begin{array}{c}\cap \\ \mathcal{P}^{Ab}(P)\end{array}$$

∘ $y(P)$ are cpt. + proj. as:

∘ $\mathrm{Hom}(yP, X) = X(P)$

∘ filt. colimits + reflex. coeq. levelwise.

$\square$

Prop: [2.54] $L : \mathcal{P}^{Set}_{\Sigma}(\Phi) \longrightarrow \mathrm{Shv}^{Set}_{\Sigma}(\Phi)$ witnesses $\mathrm{Shv}^{set}_{\Sigma}(\Phi)$ as an exact, access. localisation at $\mathcal{P}^{set}_{\Sigma}(\Phi)$, hence is also Grothendieck abelian.

$\square$

Theorem [Goerss–Hopkins]: Let $\mathcal{P} \subseteq \mathcal{A}$.

Then $y : \mathcal{A} \longrightarrow P_\Sigma^{set}(\mathcal{P})$ (restricted Yoneda)

$A \longmapsto \mathrm{Hom}_{\mathcal{A}}(-, A)$

is fully-faithful with essential image $\mathrm{Shv}_\Sigma^{set}(\mathcal{P})$.

Pf: (i) $y(A)$ is a spherical sheaf.

∘ spherical ✓ ∘ sheaf as epi's are effective. ✓

(ii) y is exact ∘ left ✓

∘ right ⇝ check y preserves epi's.

⇝ std arg. (trust me!)

(iii) y is cocontinuous ∘ y pres. filt. colimits + right exact.

(iv) y is ess. surj. onto $\mathrm{Shv}_\Sigma^{set}(\mathcal{P})$

$\mathrm{Im}(y)$ + $\mathrm{Shv}_\Sigma^{set}(\mathcal{P})$ gen. by y under colimits.

(v) y is fully-faithful

$$\mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(yA, yB)$$

WTS: is an iso.

∘ True if $A \in \mathcal{P}$ ✓,

∘ $\mathcal{A}$ is gen. by colimits.

□

In fact, $\boxed{\widehat{\mathrm{Shv}}^{Sp(p)}_{\Sigma} \simeq D(A)}$ ← This is an exercise for today!

2. Hopf algebroids and comodules

<u>Def$^{\underline{n}}$</u>: A (ring) spectrum E is of **Adams type** if $E \simeq \underset{\text{filt.}}{\operatorname{colim}} E_\alpha$ where

(i) $E_\alpha \in Sp^{fin}$, & E_*E_α is finite + proj,

(ii)

$$E^*E_\alpha \xrightarrow{\simeq} \mathrm{Hom}_{Mod_{E_*}}(E_*E^\alpha, E_*)$$

is an equivalence.

~~$H\mathbb{Z}$~~

| $\mathbb{S}$, $H\mathbb{F}_p$, ku,
| E_n, $K(n)$, MU

In this case $E_*E = \pi_*E \otimes E$ is flat over $E_* = \pi_*E$.

$\rightsquigarrow (E_*, E_*E)$ is a Hopf algebroid.

Similarly, E_*X is a comodule via:

$$\begin{tikzcd}
E_*X \arrow[r] \arrow[d, equal] & E_*X \otimes_{E_*} E_*E \arrow[d, "\simeq"] \\
\pi_*E \otimes X \arrow[r, "\mathrm{id}\otimes\mathrm{id}\otimes\eta"] & \pi_*(E \otimes X \otimes E).
\end{tikzcd}$$

Break!

Exercises:

(i) Show $\mathcal{P} \hookrightarrow \mathcal{A}$ induces an equivalence $P_\Sigma(\mathcal{P}) \xrightarrow{\simeq} D_{\geq 0}(\mathcal{A})$.

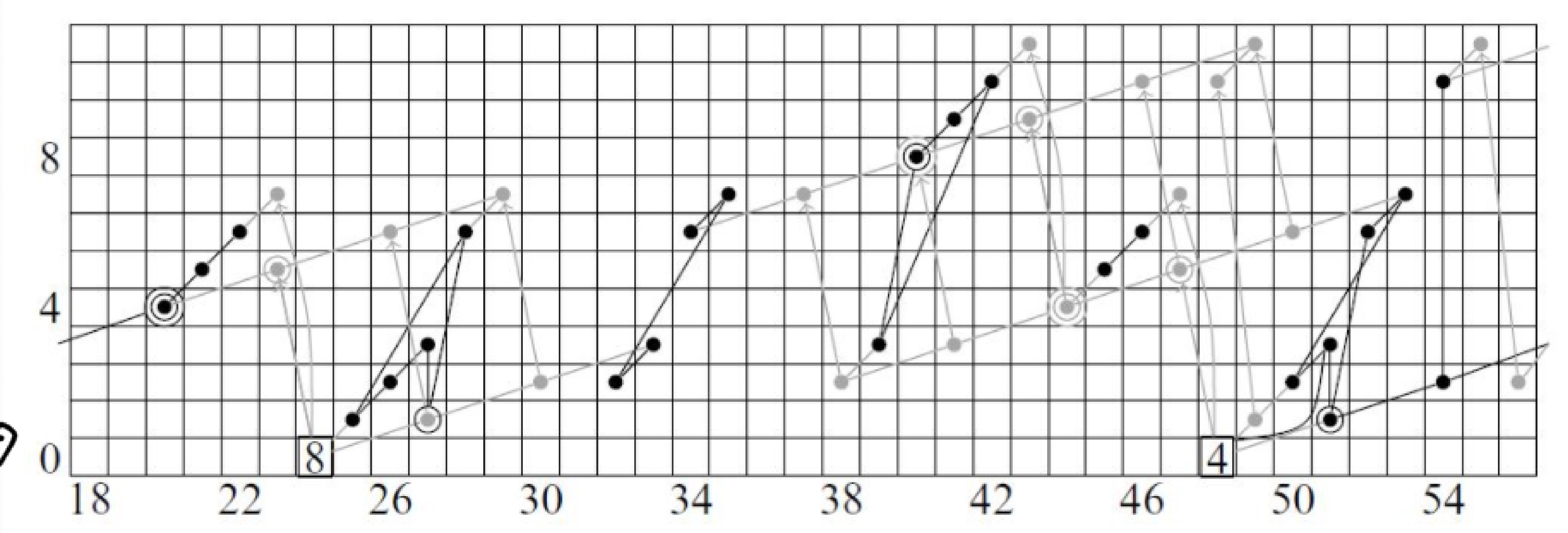
(ii) Show $\mathcal{P} \hookrightarrow \mathcal{A}$ induces an equivalence $P_\Sigma^{Sp}(\mathcal{P}) \xrightarrow{\simeq} D(\mathcal{A})$.

(iii) How does this relate to

$$\widehat{\mathrm{Shv}}_\Sigma^{Sp}(\mathcal{P}) \simeq D(\mathcal{A}).$$

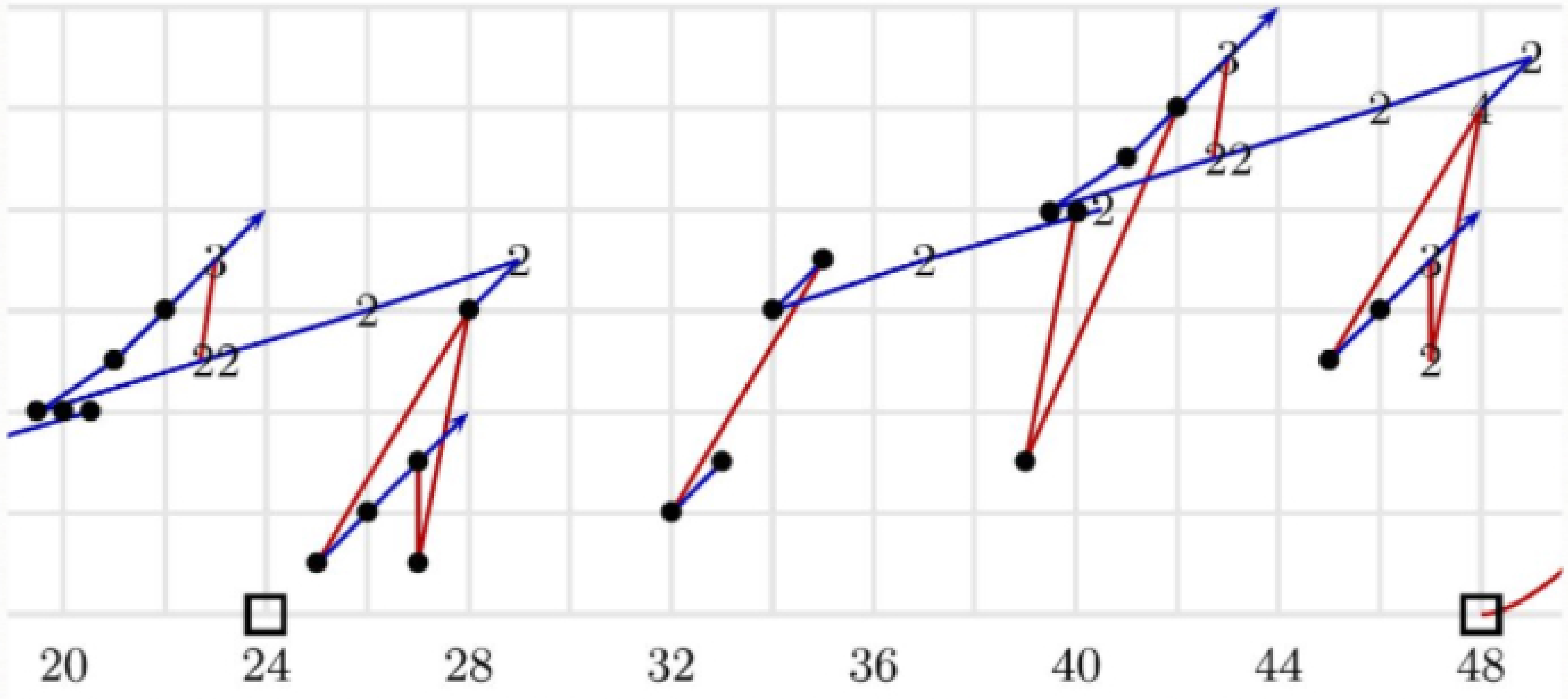
Is asking for hypercompleteness too much then?

E_∞-DSS
$tmf_{(2)}$



$\square = \hat{\mathbb{Z}}_2$
$\bullet = \mathbb{Z}/2$
$\odot = \mathbb{Z}/4$
$\circledcirc = \mathbb{Z}/8$

E_∞ - DSS
„synthetic" $tmf_{(2)}$
$= mmf.$



$\square = \mathbb{M}_2 = \hat{\mathbb{Z}}_2[\tau]$
$\bullet = \mathbb{M}_2/2$
$\bullet\bullet = \mathbb{M}_2/4$
$\bullet\bullet\bullet = \mathbb{M}_2/8$
$n = \mathbb{M}_2/\tau^n$

3. Synthetic spectra

Let E be a spectrum of Adams type.

Write $Sp_E^{fp} = \left\{ X \in Sp^{fin} \mid E_* X \text{ is fin. + proj} \right\}$.

<u>Def^{n}</u>:

$$Syn_E = Shv_\Sigma^{Sp}(Sp_E^{fp})$$

$Y \to X$ cover if $E_*Y \twoheadrightarrow E_*X$.

<u>Prop</u>: [4.2 + 4.16] Syn_E is a presentable, stable, symmetric monoidal ∞-category with t-structure.

<u>Pf</u>:

Yuqing explained this.

Furthermore,

$$Syn_E^{\heartsuit} \simeq Shv_\Sigma^{Set}(Sp_E^{fp}) \underset{\text{common envelope}}{\simeq} Shv_\Sigma^{Set}(coMod_{E_*E}^{fp}) \simeq coMod_{E_*E} \quad \square$$

Given a spectrum X, we have a presheaf yX on Sp_E^{fp}, defined by $Map_{Sp}(-,X)$.

Note that:

- Spherical ✓
- yX is a sheaf. ✓

epicovr

Indeed, given a (co)fibre seq. $F \to Q \twoheadrightarrow P$ inside Sp_E^{fp}, then this is also a (co)fibre sequence in Sp (check E_*F is fin + proj.)

so $F(P,X) \to F(Q,X) \to F(F,X)$ and its Ω^∞, $yX(P) \to yX(Q) \to yX(F)$ is a fibre sequence.

Recall the l.a. of

$$\Sigma^\infty_+ : Shv_\Sigma(Sp_E^{fp}) \underset{\leftarrow}{\overset{\perp}{\rightleftarrows}} Shv_\Sigma^{Sp}(Sp_E^{fp}) : \Omega^\infty$$

levelwise

is fully-faithful, so .....

Def^n: Define synthetic analogue

$$\nu : Sp \xrightarrow{y} Shv_\Sigma(Sp_E^{fp}) \xrightarrow{\Sigma^\infty_+} Syn_E.$$

(This is also given by (the sheafification of) $\tau_{\geq 0}F(-,X)$.)

Prop [4.4] ν has a canonical lax-symmetric monoidal structure, and preserves filtered colimits.

Pf: ∘ Σ^∞_+ cocont. + strong sym. monoidal.

∘ check $y : Sp \longrightarrow Shv_\Sigma(\underline{Sp_E^{fp}})$.

↳ filtered colimits ✓

↳ L is a l.a. and is strong sy. mon. 'mc.

↳ y is lax sym. □

$$\begin{tikzcd} Sp & Shv_\Sigma(Sp_E^{fp}) \arrow[l, bend left, "L"] \\ & Sp_E^{fp} \arrow[u, "\underline{Y}"'] \arrow[ul, bend left] \end{tikzcd}$$

[ν is **not** right exact! There is a criterion though]

Def$^{\underline{n}}$: Given $\overset{\text{top. deg}}{(t}, \overset{\text{weight}}{w)} \in \mathbb{Z}^2$, then

- $\mathbb{S}^{t,w} = \Sigma^{t-w}\nu\mathbb{S}^w$,
- $\pi_{t,w}X = \pi_0 \mathrm{Map}(\mathbb{S}^{t,w}, X)$ $\quad \forall X \in Syn_E$,
- $Y_{t,w}X = \pi_{t,w}(Y \otimes X)$ $\quad \forall X, Y \in Syn_E$.

Prop [4.11–12] Given $X \in Syn_E$, $Y \in Sp$, then $\forall (t,w)$

$\pi_{t,w}X \simeq \pi_{t-w}X(\mathbb{S}^w)$, and for $\boxed{t-w \geq 0}$,

$\pi_{t,w}\nu Y \simeq \pi_{t-w}Y$.

Pf. Yoneda etc. □

4. Hypercomplete sheaves

<u>Def$^{\underline{n}}$</u>: A sheaf of spectra $\mathcal{F}$ is ∞-connective if $\pi_n \mathcal{F} \simeq 0$, $\forall n \in \mathbb{Z}$.

We say a sheaf of spectra $\mathcal{G}$ is hypercomplete if $\forall$ ∞-conn. Y,

$$\mathrm{Map}_{\mathrm{Shv}}(\mathcal{F}, \mathcal{G}) \simeq *.$$

<u>Note</u>: $\pi_n X \neq \pi_{s,w} X$, $X \in \mathrm{Syn}_E$.
(s related)

<u>Theorem</u> [5.4] Given $X \in \mathrm{Syn}_E$, then X is hypercomplete iff X is νE-local.

<u>Pf</u>: ($\Rightarrow$) X is νE-local if $\forall Y \in \mathrm{Syn}_E$ with $\nu E_{**} Y = 0$ $\rightsquigarrow$ Y is ∞-conn.
(Tommy)
~~[illegible]~~ if $\mathrm{Map}(Y, X) \simeq *$.

$\rightsquigarrow$ if X hypercomplete $\Rightarrow$ νE-local.

($\Leftarrow$) If X is νE-local, then

$$X \xrightarrow{f} \hat{L}X, \text{ also in } \mathrm{Syn}_E.$$

hyper-sheafification

$\rightsquigarrow$ $\nu E_{**} X \xrightarrow{\simeq} \nu E_{**} \hat{L} X$ is iso.

$\Rightarrow$ f is an equiv. $\square$


Fin