Synthetic Seminar

15-2-2021

Goal: (conjecture Galatius & Randal-Williams)

Thm 1.4: If $n > 31$,

$$\pi_{8n-1}\mathbb{S} \longrightarrow \pi_{8n-1} MO\langle 4n\rangle$$

is surjective, with kernel J_{8n-1}.

approach:

trivial $\qquad J^{st}_+$

$$MO\langle 4n\rangle \simeq \mathrm{Bar}(\mathbb{S}, \Sigma^\infty_+ O\langle 4n-1\rangle, \mathbb{S})$$

last time:

Thm 4.11: The image of

$$\pi_\ell(\Sigma^\infty O\langle 4n-1\rangle) \xrightarrow{J^{st}} \pi_\ell \mathbb{S}$$

agrees with usual J_ℓ for $4n-1 \le \ell \le 8n-1$.

understanding $\mathrm{Bar}(- \wedge -)$ in dimensions $\leq 12n-2$:

define P by cofiber sequence

$$\begin{tikzcd}
\Sigma^\infty O\langle 4n-1\rangle^{\otimes 2} \arrow[r, "1\otimes J^{st} - m"] & \Sigma^\infty O\langle 4n-1\rangle \arrow[r] \arrow[d, "J^{st}"] & P \arrow[dl, dashed] \\
& \mathbb{S} &
\end{tikzcd}$$

set $C := \mathrm{cof}(P \to \mathbb{S})$

$(C \simeq \mathrm{Bar}_{\leq 2}(- \wedge -))$

<u>Thm 5.2:</u> $\tau_{\leq 12n-2} C \cong \tau_{\leq 12n-2} MO\langle 4n\rangle$

and $\mathbb{S} \to C$ agrees with unit $\mathbb{S} \to MO\langle 4n\rangle$ in this range.

get exact sequence

$$
\pi_{8n-1} P \longrightarrow \pi_{8n-1} \mathbb{S} \xrightarrow{\text{map of Thm 1.4}} \pi_{8n-1} MO\langle 4n\rangle \longrightarrow \pi_{8n-2} P
$$

<u>Lem:</u> $\pi_{8n-2} P \cong 0$

(this immediately implies surjectivity part of Thm 1.4)

<u>Pf:</u> write $x \in \pi_{4n-1} \Sigma^\infty O\langle 4n-1\rangle \cong \mathbb{Z}$ for a generator

$$
\begin{tikzcd}
\pi_{8n-2}\left(\Sigma^\infty O\langle 4n-1\rangle^{\otimes 2}\right) \arrow[r, "1\otimes J^{st} - m"] \arrow[d, phantom, "\cong" rotate=-90] & \pi_{8n-2}\left(\Sigma^\infty O\langle 4n-1\rangle\right) \arrow[d, phantom, "\cong" rotate=-90] \arrow[dl, out=-90, in=90] \\
\mathbb{Z}\{x \otimes x\} & \mathbb{Z}/2, \text{ generated by } x^2 \\
\pi_{8n-2} P \arrow[r] & \pi_{8n-3}\left(\Sigma^\infty O\langle 4n-1\rangle^{\otimes 2}\right) \cong 0
\end{tikzcd}
$$

check: x^2 is in the image of $1 \otimes J^{st} - m$

$$(1 \otimes J^{st} - m)(x \otimes x) = \underbrace{x\, J^{st}(x)}_{\neq 0 \text{ by Lem 4.10}} - x^2$$

idea: $AF(x^2) = 1$, $AF(J^{st}(x)) > 1$

$\square$

remains to understand the image of the map

$$\pi_{8n-1} P \xrightarrow{j} \pi_{8n-1} S$$

(want this to agree with J_{8n-1})

construction of P in particular gives map

$$\pi_{8n-1} P \longrightarrow \pi_{8n-2}\left(\Sigma^\infty O\langle 4n-1\rangle^{\otimes 2}\right) \cong \mathbb{Z}\{x \otimes x\}$$

$$\Omega_{8n-1} P \longrightarrow \Omega_{8n-2}\left(\Sigma^\infty O\langle 4n-1\rangle^{\otimes 2}\right) \cong \mathbb{Z}\{x \otimes x\}$$

<u>Lem:</u> Suppose $\ell \in \Omega_{8n-1} P$ maps to $2\, x \otimes x$. If $j(\ell) \in \mathcal{J}_{8n-1}$, the Thm 1.4 follows.

<u>Pf:</u>

$$\begin{tikzcd}
 & & \mathbb{Z}\{x \otimes x\} \arrow[d, phantom, "\cong" rotate=90] \\
\Omega_{8n-1}\left(\Sigma^\infty O\langle 4n-1\rangle\right) \arrow[r] \arrow[dr, "J^{st}"] & \Omega_{8n-1} P \arrow[r, "\partial"] \arrow[d, "j"] & \Omega_{8n-2}\left(\Sigma^\infty O\langle 4n-1\rangle^{\otimes 2}\right) \\
 & \Omega_{8n-1} S &
\end{tikzcd}$$

has image of $\mathcal{J}_{8n-1}$ by Thm 4.11

what is $\operatorname{im}(\partial)$?

what is a generator for $\operatorname{im}(\partial)$?

<u>claim:</u> $2\, x \otimes x$

indeed, im (∂) is the kernel of

$$\Omega_{8n-2}\left(\Sigma^\infty O\langle 4n-1\rangle^{\otimes 2}\right) \xrightarrow{1\otimes J^{st}-m} \Omega_{8n-2}\left(\Sigma^\infty O\langle 4n-1\rangle\right)$$

$\mathbb{Z}/2\,\{x^2\}$

$\cong$

$$x\otimes x \longmapsto x^2$$

So the kernel is generated by $2x\otimes x$. $\square$

For the proof of Thm 1.4, will arrange a particular choice of such an ℓ

<u>obs</u>: specifying a lift ℓ of $2x\otimes x$ is the same as specifying a null htpy of the composite

$$\begin{tikzcd} S^{8n-2} \arrow[r, "2x\otimes x"] \arrow[rr, bend right, "0"', ""{name=B}] & \Sigma^\infty O\langle 4n-1\rangle^{\otimes 2} \arrow[r, "1\otimes J^{st}-m"] \arrow[to=B, phantom, "\Updownarrow h"] & \Sigma^\infty O\langle 4n-1\rangle \end{tikzcd}$$

can construct such h by choosing null htpies of

$$(1 \otimes J^{st}) \circ (2x \otimes x) = 2x\, J^{st}(x) \qquad (1)$$

and

$$m \circ (2x \otimes x) = 2x^2 \qquad (2)$$

and then subtracting these two.

In terms of these two htpies, the class $j(\ell)$ will be defined by

$$\begin{tikzcd}
S^{8n-2} \arrow[dr, "2"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "0"'] & & \\
& S^{8n-2} \arrow[r, "\times J^{st}(x)"] \arrow[d, "x^2"] \arrow[dr, "J^{st}(x^2)"] & \Sigma^\infty O\langle 4n-1\rangle \arrow[d, "J^{st}"] \\
& \Sigma^\infty O\langle 4n-1\rangle \arrow[r, "J^{st}"'] & S
\end{tikzcd}$$

(1) (2)

comparing htpies gives map $S^{8n-1} \xrightarrow{j(\ell)} S$

notation: $w := j(\ell)$

constructing ②:

come from $\mathbb{E}_\infty$-ring structure

more precisely: in the free $\mathbb{E}_\infty$-ring on a

class x of degree $4n-1$,

have $2x^2 \simeq 0$ by Koszul sign rule

picking a htpy witnessing this relation gives

a "universal" htpy witnessing $2x^2 \simeq 0$

in ~~any~~ $\mathbb{E}_\infty$-ring (e.g. $\Sigma^\infty O\langle 4n-1\rangle$)

this is natural in maps of $\mathbb{E}_\infty$-rings taking this choice for (2) simplifies the diagram defining w to

(1) from before

$$\begin{tikzcd}
S^{8n-2} \arrow[dr, "2"] \arrow[drr, bend left, "0"] \arrow[ddrr, bend right, "0"'] & & \\
& S^{8n-2} \arrow[r, "\alpha J^{st}(x)"] \arrow[dr, "J^{st}(x^2)"] \arrow[u, phantom, "\Leftrightarrow f"] \arrow[dl, phantom, "\Leftrightarrow \mathrm{can}"] & \Sigma^\infty O\langle 4n-1\rangle \arrow[d, "J^{st}"] \\
& & S
\end{tikzcd}$$

summary: any choice of null htpy

$$f : 2x J^{st}(x) \simeq 0$$

yields a choice of w

<u>key point</u>: the diagram ⊛ admits a lift to the ∞-category $\mathrm{Syn}_{\mathrm{HF}_p}$ as follows:

$$
\begin{tikzcd}[column sep=large, row sep=huge]
S^{8n-2,\,8n+2N_p-2} \arrow[r, "2"] \arrow[rr, bend left=30, "0", ""{name=Z, below}] \arrow[drr, bend right=20, "0"'] & S^{8n-2,\,8n+2N_p-2} \arrow[r, "v(x)\,y"] \arrow[dr, "y^2"] \arrow[d, phantom, "\Updownarrow\ \mathrm{can}"] \arrow[u, phantom, "\Updownarrow\ \tilde f"] & \Sigma^{0,N_p}\, \nu\, \Sigma^\infty O\langle 4n-1\rangle \arrow[d, "\tilde J^{\mathrm{st}}"] \\
 & & S^{0,0}
\end{tikzcd}
$$

can from E_∞-structure of $S^{0,0}$

$\tilde f$ exists because

$$\pi_{8n-2,\,8n+N_p-2}\left(\nu\, \Sigma^\infty O\langle 4n-1\rangle\right) \simeq 0$$

(Prop. 10.7)

Consequence: (Thm 10.8)

w arises by applying τ^{-1} to a map

$$S^{8n-1,\, 8n+2N_p-2} \longrightarrow S^{0,0}$$

$\Rightarrow$ thus w has $H\mathbb{F}_p$-Adams filtration at least

$$(8n+2N_p-2)-(8n-1) = 2N_p - 1$$

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finishing the proof of Thm 1.4

Thm 7.1: $w \in \Omega_{8n-1} S$ is in J_{8n-1} p-locally &

* $p > 3$
* $p = 3$ & $n \geq 32$
* $p = 2$ & $n \geq 17$

(note: Thm 1.4 follows)

method: to put a lower bound on $AF(w)$ (this is $\geq 2N_p - 1$ as just discussed)

to put a upper bound on $AF(\operatorname{coker} J)$

Def: $N_2 = h(4n-1) - \lfloor \log_2 (8n) \rfloor + 1$

with $h(k) = \#\{0 < s \leq k \mid s \equiv 0, 1, 2, 4 \mod 8\}$

p odd: $N_p = \left\lfloor \frac{4n}{2p-2} \right\rfloor - \lfloor \log_p (4n) \rfloor$

upper bounds for AF(coker J)

p=2:

Thm (Davis-Mahowald)

If $\alpha \in \pi_{8n-1} S_{(2)}$ is in coker(J) and has Adams filtration

$$\geq \frac{3}{10}(8n-1) + 7 + \nu_2(n),$$

2-adic valuation

then $\alpha = 0$.

p odd:

Thm (Gonzalez)

If $\alpha \in \pi_{8n-1} S_{(p)}$ is in coker J and

$$AF(\alpha) \geq 3 + \frac{(2p-1)(8n-1)}{(2p-2)(p^2-p-1)},$$

then $\alpha = 0$.

at $p=3$, also need:

Thm (Burklund)

If $\alpha \in \pi_{8n-1} S_{(3)}$ is in $\operatorname{coker}(J)$ and

$$AF(\alpha) > \frac{25(8n-1)}{184} + 19 + \frac{1133}{1472},$$

then $\alpha = 0$.

Conclusion from these estimates:

w is p-locally trivial in $\operatorname{coker}(J)$ if

$p=2$, $\quad n \geq 17$

$p=3$, $\quad n \geq 32$

$p=5$, $\quad n \geq 16$

$p=7$, $\quad n \geq 21$

$p \geq 11$, $\quad n \geq 2(2p-2)$

To prove Thm 7.1:

need $\pi_{8n-1} S_{(p)} = J_{8n-1}$ for $p \geq 5$

and n below the bounds indicated in last

above

for $p = 5, 7, 11, 13$ this follows from

calculations $\pi_{*} S_{(p)}$ (Ravenel)

$p \geq 17$:

coker J starts with β_1 in

$(2p^2 - 2p - 2)$-stem

observe: $8n - 1 < 16(2p-2) - 1 < 2p^2 - 2p - 2$

$\square$