

# Syntetiz Seminar

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15-2-2021

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Goal: (conjecture Galatas & Radat-Williams)

Thm 1.4: If  $n > 31$ ,

$$\pi_{8n-1} \mathbb{S} \longrightarrow \pi_{8n-1} \pi_0 \langle 4n \rangle$$

is surjective, with kernel  $\int_{8n-1}$ .

approach:

$$\pi_0 \langle 4n \rangle \cong \text{Bar}(\mathbb{S}, \overset{\text{kernel}}{\sum_{l=0}^{\infty} \pi_0 \langle 4n-1 \rangle}, \mathbb{S})$$

$\int_{sb}$

last time:

Thm 4.11: The image of  $\pi_l(\sum_{i=0}^{\infty} \pi_0 \langle 4n-1 \rangle) \xrightarrow{\int_{sb}} \pi_l \mathbb{S}$

agrees with usual  $\int_l$  for  $4n-1 \leq l \leq 8n-1$ .

understanding  $\text{Bar}(-n-)$  in  
 dimensions  $\leq 12n-2$ :

define  $P$  by cotilber sequence

$$\begin{array}{ccc} \Sigma^\infty O(4n-1) \otimes \mathbb{Z} & \xrightarrow{10J-m} & \Sigma^\infty O(4n-1) \rightarrow P \\ & & \downarrow J^{\text{ct}} \\ & & \mathbb{Z} \end{array}$$

set  $C := \text{col}(P \rightarrow \mathbb{Z})$

$$(C \cong \text{Bar}_{\leq 2}(-n-))$$

Thm 5.2:  $\tau_{\leq 12n-2} C \cong \tau_{\leq 12n-2} \mathbb{N}O(4n)$

and  $\mathbb{Z} \rightarrow C$  agrees with unit  $\mathbb{Z} \rightarrow \mathbb{N}O(4n)$   
 in this range.

get exact sequence *map of Thm 1.4*

$$\pi_{8n-1} P \rightarrow \pi_{8n-1} S \xrightarrow{\text{map}} \pi_{8n-1} MO(4n) \rightarrow \pi_{8n-2} P$$

Lemma:  $\pi_{8n-2} P \cong 0$

(this immediately implies surjectivity part of Thm 1.4)

Pf: write  $x \in \pi_{4n-1} \Sigma^\infty O(4n-1) \cong \mathbb{Z}$

for a generator

$$\pi_{8n-2} (\Sigma^\infty O(4n-1)^{\otimes 2}) \xrightarrow{\text{J-h}} \pi_{8n-2} (\Sigma^\infty O(4n-1))$$

$\cong \mathbb{Z} \{x \otimes x\}$  
 $\cong \mathbb{Z}/2$ , generated by  $x^2$

$$\pi_{8n-2} P \xrightarrow{\cong} \pi_{8n-3} (\Sigma^\infty O(4n-1)^{\otimes 2})$$

check:  $x^2$  is in the image of  $1 \otimes J_{-m}^{st}$

$$(1 \otimes J_{-m}^{st})(x \otimes x) = x \underbrace{J_{-m}^{st}(x)} - x^2$$

$\square 0$  by Lem 4.10

iden:  $AF(x^2) = 1$ ,  $AF(J_{-m}^{st}(x)) > 1$

$\square$

remains to understand the image of the map

$$\pi_{\mathcal{B}_{n-1}} \mathcal{P} \xrightarrow{j} \pi_{\mathcal{B}_{n-1}} \mathcal{S}$$

(want this to agree with  $\int_{\mathcal{B}_{n-1}}$ )

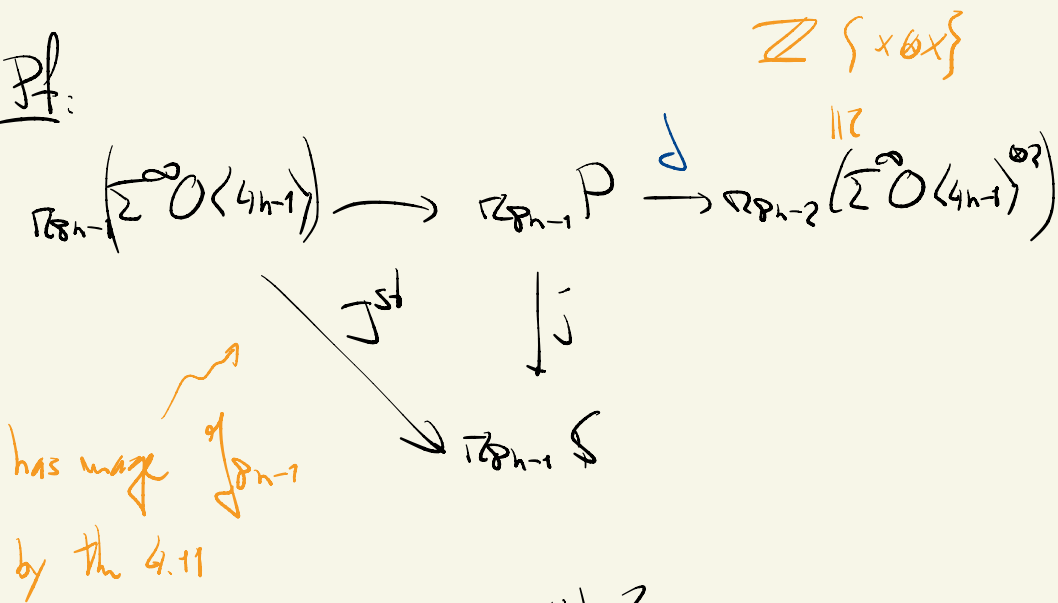
construction of  $\mathcal{P}$  in particular gives map

$$\pi_{\mathcal{B}_{n-1}} \mathcal{P} \longrightarrow \pi_{\mathcal{B}_{n-2}} \left( \sum_{\infty} \mathbb{O} \leq 4n-1 \right)^{\otimes 2} \cong \mathbb{Z} \langle x \otimes x \rangle$$

$$\Omega_{\mathbb{P}^{n-1}} \longrightarrow \Omega_{\mathbb{P}^{n-2}} \left( \sum_{i=0}^{\infty} \mathcal{O}(4n-1)^{\otimes 2} \right) \cong \mathbb{Z} \{x \otimes x\}$$

Lemma: Suppose  $l \in \Omega_{\mathbb{P}^{n-1}} \mathbb{P}$  maps to  $2x \otimes x$ . If  $j(l) \in \mathcal{I}_{\mathbb{P}^{n-1}}$ , then the 1.4 follows.

Pf:



what is  $m(\mathcal{J})$ ?

what is a generator for  $m(\mathcal{J})$ ?

claim:  $2x \otimes x$

indeed,  $\text{im } \Delta \subseteq$  the kernel of  $\mathbb{Z}/2 \langle x^2 \rangle$   
 $\mathbb{Z}_{2n-2} \left( \sum_{i=0}^{\infty} O(4n-1)^{\otimes 2} \right) \xrightarrow{1 \otimes J_{-m}^{\text{st}}} \mathbb{Z}_{2n-2} \left( \sum_{i=0}^{\infty} O(4n-1) \right)$   
 $x \otimes x \quad \longmapsto \quad x^2$

so the kernel  $\subseteq$  generated by  $2x \otimes x$ .  $\square$

for the proof of Thm 1.4, will arrange a particular choice of such an  $l$

obs: specifying a lift  $l$  of  $2x \otimes x \subseteq$  the same as specifying a null htpy of the composite

$$\mathbb{Z}_{2n-2} \xrightarrow{2x \otimes x} \sum_{i=0}^{\infty} O(4n-1)^{\otimes 2} \xrightarrow{1 \otimes J_{-m}^{\text{st}}} \sum_{i=0}^{\infty} O(4n-1)$$

$\Downarrow h$

0

can construct such  $h$  by choosing null hypotheses

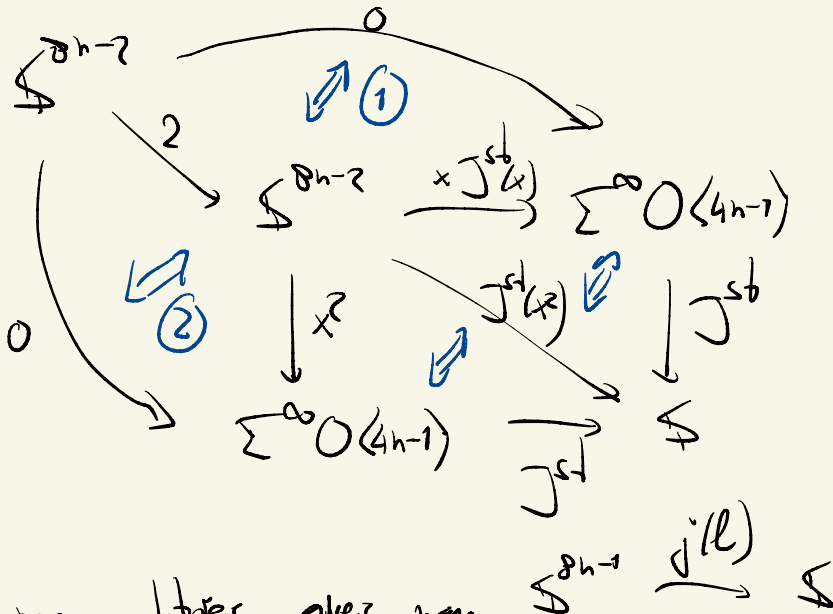
$$(1 \otimes J^{st}) \circ (2 \times \otimes x) = 2 \times J^{st}(x) \quad (1)$$

$$\text{and } m \circ (2 \times \otimes x) = 2x^2 \quad (2)$$

and then subtracting these two.

in terms of these two hypotheses, the class  $j(l)$

will be defined by



comparing hypotheses gives map



notation:  $w_i = j(l)$

Constructing  $\textcircled{?}$ :

come from  $\mathbb{Z}_2$ -ring structure

more precisely: in the free  $\mathbb{Z}_2$ -ring on a

class  $x$  of degree  $4n-1$ ,

have  $2x^2 \simeq 0$  by Koszul sign rule

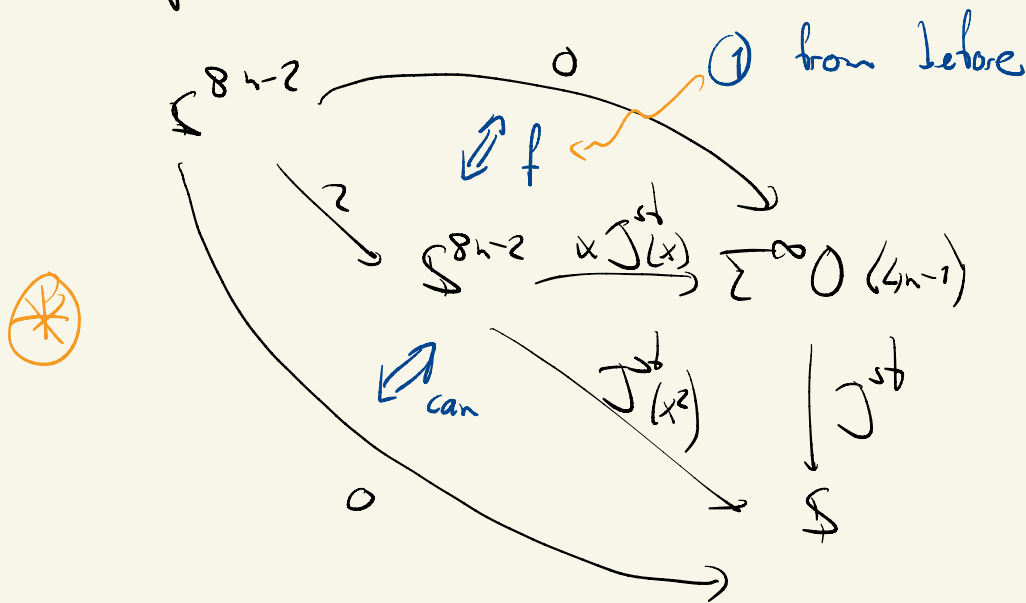
proving a htopy witnessing this relation gives

a "universal" htopy witnessing  $2x^2 \simeq 0$

in any  $\mathbb{Z}_2$ -ring (e.g.  $\Sigma^\infty \mathbb{O}(4n-1)$ )

this is natural in maps of  $\mathbb{Z}_2$ -rings

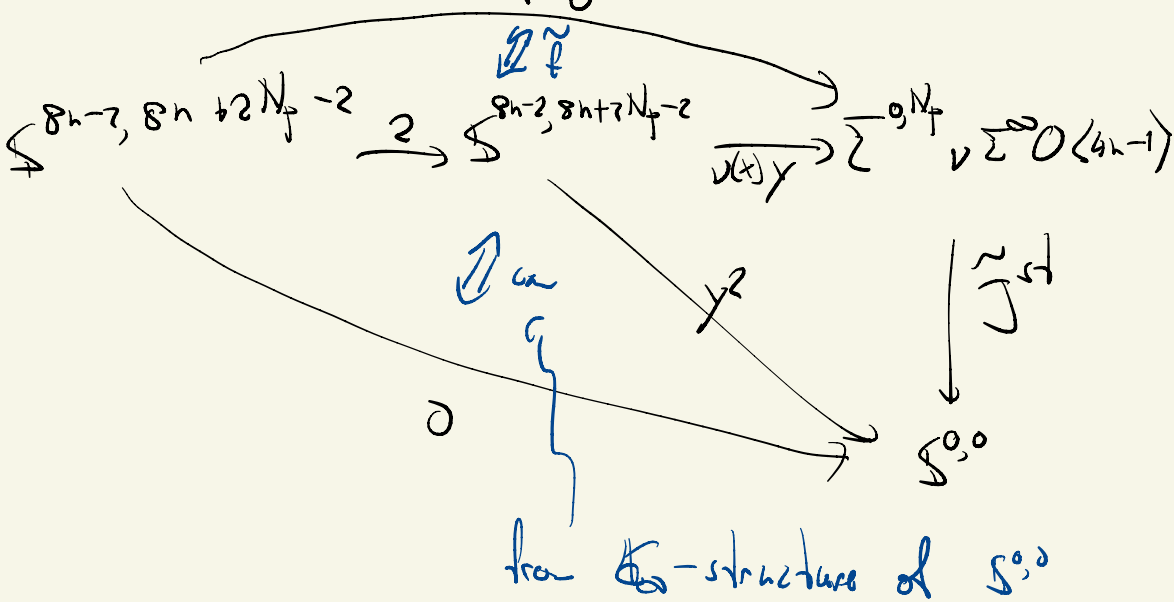
taking this choice for  $\textcircled{2}$  simplifies the diagram defining  $w$  to



summary: ~~any~~ choice of null hypothesis

$f: 2x J^{st}(x) \leq 0$   
yields a choice of  $w$

key point: the diagram  $\circledast$  admits a lift to the  $\infty$ -category  $\text{Syn}_{\text{HTF}_p}$  as follows:



$\tilde{f}$  exists because

$$S^{8n-2, 8n+2N_p-2} (\nu \Sigma^\infty O\langle 4n-1 \rangle) \simeq 0 \quad (\text{Prop 10.7})$$

Consequence: (Th 10.8)

$w$  arises by applying  $\tau^{-1}$  to a map

$$\int_{\partial^{n-1}, \partial^n + 2N_p - 2} \rightarrow \int_{0,0}$$

$\Rightarrow$  this  $w$  has  $H\mathbb{F}_p$ -Atlas filtration  
at least

$$(\partial^n + 2N_p - 2) - (\partial^{n-1}) = 2N_p - 1$$

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finishing the proof of Th 1.4

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Th 7.1:  $w \in \Omega_{\partial^{n-1}} \mathcal{S}$  is in  $\int_{\partial^{n-1}}$   
 $p$ -locally  $\mathcal{S}$

$$* \quad p > 3$$

$$* \quad p = 3 \quad \& \quad n \geq 32$$

$$* \quad p = 2 \quad \& \quad n \geq 17$$

(note: The 1.4 follows)

method:  $\rightarrow$  put a lower bound on  
 $AF(w)$  ( $4w$  is  $\geq 2N_p - 1$   
 as just discussed)

$\rightarrow$  put an upper bound on  
 $AF(\text{coker } J)$

$$\underline{\text{Def.}} \quad N_2 = h(4n-1) - \lfloor \log_2(8n) \rfloor + 1$$

$$\text{with } h(k) = \# \{ 0 < s \leq k \mid s \equiv 0, 1, 3, 4 \pmod{8} \}$$

$$\underline{p \text{ odd:}} \quad N_p = \left\lfloor \frac{4n}{2p-2} \right\rfloor - \lfloor \log_p(4n) \rfloor$$

upper bounds for AF (coker  $J$ )

$p=3$ :

Thm (Davis-Mahowald)

If  $\alpha \in \pi_{2n-1} S_{(2)} \cong \pi_{2n-1} \text{coker}(J)$

and has Adams filtration

$$\geq \frac{3}{10} (8n-1) + 1 + \frac{1}{2} (n),$$

$\swarrow$   
2-adic valuation

then  $\alpha = 0$ .

$p$  odd:

Thm (Gonzalez)

If  $\alpha \in \pi_{2n-1} S_{(p)} \cong \pi_{2n-1} \text{coker}(J)$  and

$$\text{AF}(\alpha) \geq 3 + \frac{(2p-1)(8n-1)}{(2p-2)(p^2-p-1)},$$

then  $\alpha = 0$ .

at  $p=3$ , also need:

Thm (Birkland)

If  $\alpha \in \Omega_{p, h-1} \setminus \binom{[p]}{2}$  is in  $\text{coker}(J)$  of

$$AF(\alpha) > \frac{25(p-1)}{184} + 19 + \frac{1133}{1472},$$

then  $\alpha = 0$ .

Conclude from these estimates:

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$w$  is  $p$ -locally trivial in  $\text{coker}(J)$  if

$$p=2, \quad n \geq 17$$

$$p=3, \quad n \geq 32$$

$$p=5, \quad n \geq 16$$

$$p=7, \quad n \geq 21$$

$$p \geq 11, \quad n \geq 2(2p-2)$$

To prove Thm 7.1:

need  $\tau_{2n-1} S(p) = \int p_{2n-1}$  for  $p \geq 5$

and  $n$  below the bounds indicated in list above

for  $p = 5, 7, 11, 13$  this follows from calculations  $\tau_{2n} S(p)$  (Ravneil)

$p \geq 17$ :

coke  $\mathcal{J}$  starts with  $\beta_1$  in  $(2p^2 - 2p - 2)$ -stem

observe:  $8n-1 < 16(2p-2)-1 < 2p^2 - 2p - 2$

□