Synthetic spectra via motivic homotopy theory (after Gheorghe, Isaksen, Pstrągowski, ...)

I. Motivic homotopy theory

"homotopy theory of schemes"

Construction: S_k^{mot}

1) Start with Sm_k (smooth schemes / k).

2) $Shv_{Nis}(Sm_k, S)$ — $Y \mapsto Sm_k$ — has the effect of "freely adjoining homotopy colimits", gluing those that come from covers:

$colim(\ldots) \simeq Y(X)$.

$U \cap V \to V$, $U \to X$ $\rightsquigarrow$ $Y(U \cup V) \to Y(V)$, $Y(U) \to Y(X)$

3) Localize $Shv(Sm_k, S)$ at all maps $A^1 \times X \to X$.

2 spheres: $S^{1,0}$ "topological sphere": $pt \sqcup pt \to pt$, $pt \to S^{1,0}$

$S^{1,1}$ "algebraic sphere": $A^1 \setminus 0$.

Form $\wedge$ ... $X \vee Y \to X \times Y$, $* \to X \wedge Y$

dimension, weight

$S^{a,w} := \bigwedge^{a-w} S^{1,0} \wedge \bigwedge^{w} S^{1,1}$

Example: P^1: $A^1 \setminus 0 \to A^1$, $A^1 \to P^1$; $S^{1,1} \to pt$, $pt \to P^1$

P^1 has a "CW structure" with one $(0,0)$-cell, $(2,1)$-cell.

P^n has a CW structure with one $(2k,k)$-cell for each $0 \le k \le n$.

Specialize $k = \mathbb{C}$:

Have Betti: $S_{\mathbb{C}}^{mot} \to S$, $X \mapsto X(\mathbb{C})$

$P^n \mapsto \mathbb{C}P^n$

$S^{a,w} \mapsto S^a$

Remark: Not any object is "cellular". We could localize wrt. $\pi_{*,*}$-isos, "cellular objects".

Def: $Sp_{\mathbb{C}}^{mot}$ category of "motivic spectra" by stabilizing wrt. both $S^{1,0}, S^{1,1}$. $S^{2,1} \simeq P^1$

closed symmetric monoidal

- stable ∞-category.
- have $S^{a,w}$, $(a,w) \in \mathbb{Z} \times \mathbb{Z}$.

$\Rightarrow$ bigraded homotopy groups $\pi_{a,w}$, also for any $E \in Sp_{\mathbb{C}}^{mot}$ bigraded homology, cohomology theories:

$E_{a,w}(X) = \pi_{a,w}(E \otimes X)$

$E^{a,w}(X) = [X, E]_{-a,-w}$

Betti realization gives $Sp_{\mathbb{C}}^{mot} \to Sp$.

II. Motivic A.S.S. and A.N.S.S.

Thm (Voevodsky): There are motivic E.M. spectra $H\mathbb{F}_p^{mot}$.

$\pi_{*,*} H\mathbb{F}_p^{mot} = \mathbb{F}_p[\tau]$, $|\tau| = (0,-1)$

$\pi_{*,*}(H\mathbb{F}_p^{mot} \otimes H\mathbb{F}_p^{mot}) = A_{*,*}^{mot, p}$

$= \mathbb{F}_p[\tau, \tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]$ (classical) $/ (\tau_i^2 = 0$ p odd, or $\tau_i^2 = \tau\xi_{i+1}$ $p = 2)$

$|\tau| = (0,-1)$

$|\tau_i| = (2p^i - 1, p^i - 1)$

$|\xi_i| = (2p^i - 2, p^i - 1)$

weight half degree (rounded down)

Lemma: $(H\mathbb{F}_{p\,*,*}^{mot} X)[\tau^{-1}]_{\tau=1} = H\mathbb{F}_{p*} Betti(X)$

There is a motivic A.S.S. (based on $H\mathbb{F}_p^{mot}$). Inverting τ on the motivic A.S.S. for X yields the classical A.S.S. for $Betti(X)$.

(motivic A.S.S. "=" classical A.S.S. + weight + τ-torsion.)

Corollary: $\tau \in \pi_{0,-1}(S^{mot})_p^\wedge$

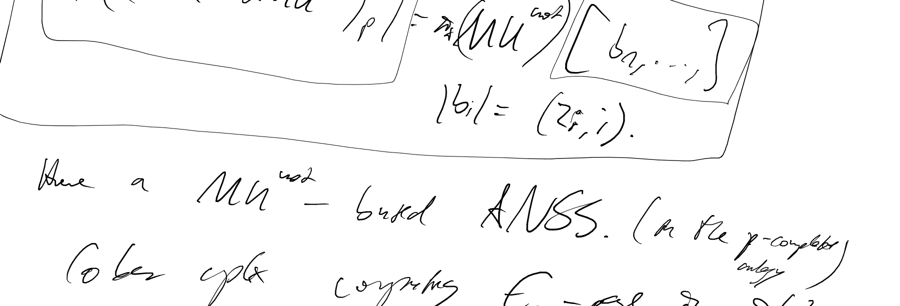
$\pi_{*,*}(X)[\tau^{-1}] = \pi_*(Betti(X))$, X p-complete.

Similarly, have spectra MGL or MU^{mot}

$\pi_{*,*}(MU^{mot})_p^\wedge = \mathbb{Z}_p[\tau, x_1, \ldots, x_i, \ldots]$, $|x_i| = (2i, i)$, $|\tau| = (0,-1)$

$\pi_{*,*}((MU^{mot} \otimes MU^{mot})_p^\wedge) = \pi_*(MU^{mot})[b_1, \ldots]$, $|b_i| = (2i, i)$

Have a MU^{mot}-based ANSS. (For the p-complete sphere.) Its E_2-page is obtained from the classical one by "weight = degree/2", adjoining τ.



chromatic (bottom):

- classical ANSS detects the bottom row.
- More is ... "multiplication by τ", so τ-... $\pi_{*,*}(S^{mot})$ corresponds ... of length ... (in classical ANSS).
- Mod τ: everything degenerates so ... classical ANSS E_2-page.
- ... motivic ANSS ... with the "τ-Bockstein S.S." for S^{mot}.

Question: ... classical homotopy groups can be "inflated" ... (with ...).

Filtered spectra:

Def: $Fil(Sp) = Fun(\mathbb{Z}^{op}, Sp)$

$\to X_0 \to X_1 \to X_2 \to$

bigraded homotopy groups:

$\pi_{a,w}(X_\bullet) = \pi_a(X_w)$

For the maps $S: S^{a,w} \to X$ ...

...

Filtered spectra give rise to S.S. with ...

Thm (Gheorghe-Isaksen-...): The functor $S_{\mathbb{C}}^{mot} \to Fil(Sp)$ ...

$\pi_{a,w}(S^{mot}) = \pi_{a,w}(...)$

... $S_{\mathbb{C}}^{mot}$ as modules over $\Omega_{\mathbb{C}}^{0,0}(S^{mot})$

What is this filtered spectrum?

- It is not the ... S
- Its associated graded has homotopy groups of the ANSS E_2-term.

ANSS: Write

$S = Tot(MU \rightrightarrows MU \otimes MU \ldots)$

$Tot(\ldots)$

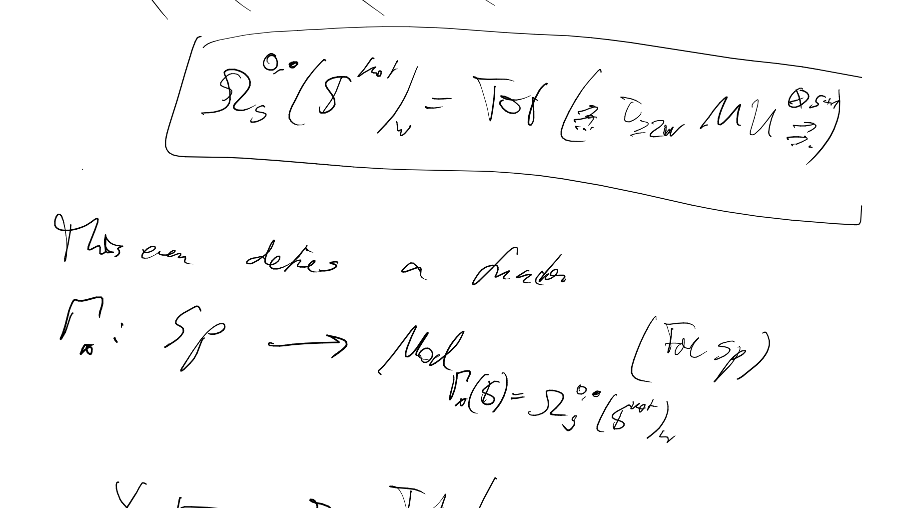
Get a filtration on S ... $S \to S \to S$ ...

Associated graded is "weight" ...

ANSS E_2-page.

Observation (Deligne): There is a construction $Déc: Fil\,Sp \to Fil\,Sp$ which "accelerates" the S.S. by one page.

For example, Déc of the Tot-tower we just constructed is a filtered spectrum, associated graded has homotopy groups ANSS E_2-page.



$$\Omega_S^{0,0}(S^{mot})_w = Tot(\tau_{\geq 2w} MU^{\otimes s+1})$$

This even defines a functor

$\Gamma_*: Sp \to Mod_{\Gamma_*(S) = \Omega_S^{0,0}(S^{mot})_w}(Fil\,Sp)$

$X \mapsto Tot(\tau_{\geq 2w} MU^{\otimes s+1} \otimes X)$.

$\Gamma_*(S^{2n}) = \Sigma^{2n,n} \Gamma_*(S)$

$\Gamma_*(S^{2n+1}) = \Sigma^{2n+1,n} \Gamma_*(S)$

produces the usual "motivic refinements", for $H\mathbb{F}_p$, ku, MU, ..., tmf

In general, one can build a "E-synthetic category" by taking modules in Fil Sp over déc of the E-based A.N. filtration of the sphere.

(i.e. $Tot(\tau_{\geq 2w} E^{\otimes s+1})$)

O:

$Fil\,Sp \xrightarrow{\tau_{\geq 0}^{Bei}} BFil\,Sp \xrightarrow[\text{the spectral sequence}]{\text{Colim}} Fil\,Sp$.