Properties of synthetic spectra

0. Recollections

Definition Let E be an Adams type homology theory.

A synthetic spectrum based on E is a spherical sheaf of spectra on E-finite spectra:

$$\mathrm{Syn}_E := \mathrm{Shv}^{Sp}_{\Sigma}(Sp^{fp}_E).$$

The category Syn_E is presentable, stable and inherits a symm. mon. structure from Sp^{fp}_E.

The synthetic analogue $\nu : Sp \to \mathrm{Syn}_E$

There is an equivalence

$$\begin{tikzcd} \mathrm{Sh}_{\Sigma}(Sp^{fp}_E) \arrow[r, "\simeq"] \arrow[dr, "\wr"'] & \mathrm{Sh}^{Sp}_{\Sigma}(Sp^{fp}_E)_{\geq 0} \arrow[d, "\wr"] \\ & \text{"}\Sigma^{\infty}_{+}\text{"}\,\mathrm{Sh}_{\Sigma}(Sp^{fp}_E) \end{tikzcd}$$

Recall that $X \in P\mathrm{Shv}_{\Sigma}(\mathcal{C})$ factors as

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r] \arrow[dr] & \mathcal{S}_* \\ & \mathrm{Alg}^{gp}_{\mathbb{E}_\infty}(\mathcal{S}_*) \arrow[u] \end{tikzcd}$$

Definition $X \in Sp$. The synthetic analogue $\nu X \in \mathrm{Syn}_E$ is the unique connective synthetic spectrum associated to the Yoneda embedding $y(X)$:

$$\mathrm{Sh}_\Sigma(\mathrm{Sp}_E^{fp}) \simeq \mathrm{Sh}_\Sigma^{\mathrm{Sp}}(\mathrm{Sp}_E^{fp})_{\geq 0} \subseteq \mathrm{Syn}_E$$

$$y(X), \longrightarrow \nu X$$

$$P \longmapsto \mathrm{map}(P, X)$$

Alternatively, νX arises as the sheafification of the presheaf $P \mapsto F(P, X)_{\geq 0}$

function spectrum

Definition

1) $S^{t,w} := \Sigma^{t-w} \nu S^w$

top degree

weight

2) $t-w :=$ Chow degree.

3) $Y_{t,w} X := \pi_0 \mathrm{map}(S^{t,w}, Y \otimes X) = \pi_{t,w}(Y \otimes X)$

Outline

1) Explain how connectivity in Syn_E is governed by E-homology.

2) Construct $\tau \in \pi_{0,-1} S^{0,0}$, and prove $\mathrm{Syn}_E(\tau^{-1}) \simeq \mathrm{Sp}$.

3) Endow $C\tau$ with a comm. alg. structure and prove $\mathrm{Mod}_{C\tau} \simeq \mathrm{Stable}_{E_*E}$ in good cases.

1. The functor $(\nu E)_{*,*}$

Lemma $P \in \mathrm{Sp}_E^{fp}$, $X \in \mathrm{Syn}_E$.

1) $\mathrm{map}(\nu P, X) \simeq \Omega^\infty X(P)$

2) $\pi_{t,w} X \simeq \pi_{t-w} X(S^w)$.

Proof 1) $\mathrm{map}(\nu P, X) = \mathrm{map}(\text{"}\Sigma^\infty_+\text{"} y(P), X)$

$\simeq \mathrm{map}(y(P), \Omega^\infty X)$

$\simeq \Omega^\infty X(P)$.

2) Exercise.

Corollary $\pi_{t,w}(\nu X) \cong \pi_t(X)$ for $t-w \geq 0$.

Proof $\pi_{t,w}(\nu X) \cong \pi_{t-w} \nu X(S^w)$

$\cong \pi_{t-w}\, y(X)(S^w) \cong \pi_{t-w} \mathrm{map}(S^w, X)$. ◪

Proposition Syn_E has a right complete t-structure, and $\mathrm{Syn}_E^\heartsuit \simeq \mathrm{Comod}_{E_*E}$.

Lemma $X \in \mathrm{Syn}_E$.

$$(\pi_n^\heartsuit X)_e = \varinjlim \pi_n X(\Sigma^e DE_\alpha),$$

$\cap$ Comod_{E_*E}

where $\varinjlim E_\alpha \simeq E$.

fin. projective

Proof Omitted, but uses explicit equivalence

$\mathrm{Comod}_{E_*E} \xrightarrow{\simeq} \mathrm{Syn}_E^\heartsuit$.

Theorem $X \in Sp_{nE}$.

$$(\pi_k^\heartsuit X)_e \simeq (\nu E)_{k+e,e}(X)$$

Proof By suspending, reduce to $k=0$.

$$(\pi_0^\heartsuit X)_e = \varinjlim \pi_0 X(\Sigma^e DE_\alpha)$$

$(\pi_k^\heartsuit X)_e = \varinjlim \pi_k X(\Sigma^e DE_\alpha)$

$$= \varinjlim \pi_0 \Omega^\infty X(\Sigma^e DE_\alpha)$$

$$\simeq \varinjlim \pi_0 \operatorname{map}(\nu \Sigma^e DE_\alpha, X)$$

$\operatorname{map}(\nu P, X) \simeq \Omega^\infty X(P)$

$$\simeq \varinjlim \pi_0 \operatorname{map}(\nu S^e \otimes \nu DE_\alpha, X)$$

ν symm. monoidal on finite spectra

$$\simeq \varinjlim \pi_0 \operatorname{map}(\nu S^e, [\nu DE_\alpha, X])$$

$$\simeq \varinjlim \pi_0 \operatorname{map}(\nu S^e, [\nu DE_\alpha, S^{0,0}] \otimes X)$$

Strong dualizability of compacts.

$$\simeq \varinjlim \pi_0 \operatorname{map}(\nu S^e, \nu E_\alpha \otimes X)$$

$$\simeq \pi_0 \operatorname{map}(\nu S^e, \varinjlim(\nu E_\alpha) \otimes X) \simeq \pi_0 \operatorname{map}(S^{e,e}, \nu E \otimes X)$$

$$= (\nu E)_{e,e}(X).$$

Corollary $X \in (Sp_{nE})_{\geq 0} \iff (\nu E)_{t,w}(X) = 0 \quad \forall\, t-w < 0.$

Corollary $X \in Sp$.

$$(\nu E)_{t,w}(\nu X) \simeq \begin{cases} E_t(X), & t-w \geq 0, \\ 0, & \text{else}. \end{cases}$$

Proof $\pi_{t,w}(\nu E \otimes \nu X)$

$\overset{!!}{\simeq} \pi_{t,w}(\nu(E \otimes X))$

$\simeq \pi_t(E \otimes X).$

Lemma $(F \to Y \to X) \in Sp$ fiber seq.

$(\nu F \to \nu Y \to \nu X) \in Syn_E$ fiber seq.

$\iff$ $0 \to E_*(F) \to E_*(Y) \to E_*(X) \to 0$ short exact.

Proof $(y(F) \to y(Y) \to y(X)) \in Sh_\Sigma(Sp_E^{fp})$ fiber seq

$\Rightarrow$ $(\nu F \to \nu Y \to \nu X) \in (Syn_E)_{\geq 0}$ fiber seq.

If $G := \mathrm{fib}_{Syn_E}(\nu Y \to \nu X)$, want $G \in (Syn_E)_{\geq 0}$.

$$\begin{tikzcd}
\pi_0^{\heartsuit} \nu Y \arrow[r, phantom, "\cong"] \arrow[d] & E_*(Y) \arrow[d] \\
\pi_0^{\heartsuit} \nu X \arrow[r, phantom, "\cong"] \arrow[d] & E_*(X) \\
\pi_{-1}^{\heartsuit} G &
\end{tikzcd}$$

Exercises

1) Let $X \in Syn_E$. Show that

$$\pi_{t,w} X \simeq \pi_t X(S^w).$$

(Use $\mathrm{map}(\nu P, X) \simeq \Omega^\infty X(P)$).

2) There are eq.'s

$$Syn_E^{\heartsuit} \simeq (Shv_\Sigma^{Sp}(Sp_E^{fp}))^{\heartsuit} \simeq Shv_\Sigma^{Ab}(Sp_E^{fp}).$$

$\Rightarrow$ Show that

$$\mathrm{Comod}_{E_*E} \xrightarrow{\simeq} Shv_\Sigma^{Ab}(Sp_E^{fp})$$
$$M \mapsto \mathrm{Hom}_{E_*E}(E_*(-), M)$$

3) For formal reasons,

$\Sigma^{h,0} \nu P \in Syn_E$ are compact for $P \in Sp_E^{fp}$.

Show that

$\{\Sigma^{h,0} \nu P \mid P \in Sp_E^{fp}\}$ generate Syn_E under colimits, so that Syn_E is cptly generated.

2. The map γ

Definition

$$\begin{tikzcd} \nu(\Omega S^0) \arrow[r, "\gamma"] \arrow[d, equal] & \Omega \nu S^0 \arrow[d, equal] \\ \nu(S^{-1}) \arrow[d, equal] & \Omega(S^{0,0}) \arrow[d, equal] \\ S^{-1,-1} & S^{-1,0} \end{tikzcd}$$

so $\gamma \in \pi_{0,-1}(S^{0,0})$.

Proposition $X \in \mathrm{Syn}_E$.

(1) $(\Sigma^{h,e} X)(P) \simeq \Sigma^{h-e} X(\Sigma^{-e} P)$.

(2)
$$\begin{tikzcd} (\Sigma^{-1,-1} X)(P) \arrow[r, "(\gamma \otimes X)(P)"] \arrow[d, "\wr\ (1)"] & (\Sigma^{-1,0} X)(P) \arrow[d, "\wr\ (1)"] \\ X(\Sigma P) \arrow[r, "\text{colim-to-lim}"] & \Omega X(P) \end{tikzcd}$$

commutes.

Proof (1) First reduce to $h=e$.

$$\begin{tikzcd} (\Sigma^{e,e} X)(P) \arrow[r, "??", "\simeq"'] \arrow[d, equal] & X(\Sigma^{-e} P) \arrow[dd, equal] \\ (S^{e,e} \otimes X)(P) \arrow[d, "\wr"] & \\ (\nu S^e \otimes X)(P) \arrow[r, "\simeq"] & X(S^{-e} \otimes P) \end{tikzcd}$$

Lemma 2.25

For $\mathcal{C}$ an excellent co-site, $c, d \in \mathcal{C}$, $A \in \mathrm{PSh}_\Sigma(\mathcal{C})$, $(y(c) \otimes A)(d) \simeq A(c^* \otimes d)$

(2) Spell out definitions and use (1). $\square$

Lemma Consider the cofib. sequence

$$\Sigma^{0,-1}\nu X \xrightarrow{\tau\otimes\nu X} \nu X \longrightarrow C\tau\otimes\nu X.$$

We have $C\tau\otimes\nu X = (\nu X)_{\leq 0} \in (\mathrm{Syn}_E)^{\heartsuit}$.

Proof

$$\begin{tikzcd}
\Sigma^{0,-1}\nu X \arrow[r] \arrow[d, equal] & \nu X \arrow[r] & C\tau\otimes\nu X \\
S^{0,-1}\otimes\nu X \arrow[d, equal] & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
\Sigma\nu S^{-1}\otimes\nu X \arrow[d, equal] & (\mathrm{Syn}_E)_{\geq 0} & (\mathrm{Syn}_E)_{\leq 0}. \\
\Sigma\nu(\Sigma^{-1}X) & & \\
\rotatebox{90}{$\in$} & & \\
(\mathrm{Syn}_E)_{\geq 1} & &
\end{tikzcd}$$

Apply Σ^{-1}:

$$\begin{tikzcd}
(\Sigma^{-1,-1}\nu X)(P) \arrow[r] \arrow[d, "\cong"] & (\Sigma^{-1,0}\nu X)(P) \arrow[d, "\cong"] & \\
(\nu X)(\Sigma P) \arrow[r] \arrow[d, "\wr"] & \Omega\nu X(P) \arrow[d, "\wr"] & \\
F(\Sigma P, X)_{\geq 0} \arrow[r] & \Omega F(X,P)_{\geq 0} \arrow[r] & \Omega H[P,X]. \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
(\mathrm{Syn}_E)_{\geq 0} & (\mathrm{Syn}_E)_{\geq -1} & (\mathrm{Syn}_E)_{\leq -1}
\end{tikzcd}$$

$\square$

3. τ-invertible synthetic spectra

<u>Definition</u> $X \in \mathrm{Syn}_E$ is τ-invertible

if $\Sigma^{0,-1} X \xrightarrow{\tau} X$ is an equivalence.

$\rightsquigarrow$ category $\mathrm{Syn}_E(\tau^{-1}) \underset{\longrightarrow}{\overset{\tau^{-1}}{\longleftarrow}} \mathrm{Syn}_E$,

$\tau^{-1} X \simeq X \otimes \tau^{-1} \mathbb{S}^{0,0}$,

$\tau^{-1}\mathbb{S}^{0,0}$ comm. alg. and $\mathrm{Syn}_E(\tau^{-1}) \simeq \mathrm{Mod}_{\tau^{-1}\mathbb{S}^{0,0}}$.

<u>Lemma</u> $X \in (\mathrm{Syn}_E)_{\leq h} \Rightarrow \tau^{-1} X \simeq 0$.

<u>Proof</u> $\tau^{-1} X \simeq \lim(\cdots \xrightarrow{\tau} \Sigma^{0,n} X \xrightarrow{\tau} \Sigma^{0,n-1} X \xrightarrow{\tau} \Sigma^{0,n-2} X \xrightarrow{\tau} \cdots)$

$X \in (\mathrm{Syn}_E)_{\leq h} \Rightarrow \Sigma^{0,n} X \in (\mathrm{Syn}_E)_{\leq (h-n)}$.

Done by right completeness of t-structure.

<u>Setting</u> $X \in \mathrm{Sp}$.

$Y(X) \in \mathrm{PShv}^{\mathrm{Sp}}(\mathrm{Sp}_E^{fp})$, $\quad Y(X)(P) := F(P, X)$.

- $Y(X)$ is a spherical sheaf (so in Syn_E).

- Since νX arises from sheafifying $P \mapsto F(P,X)_{\geq 0}$, have a map $\nu X \to Y(X)$.

Proposition $X \in Sp$.

1) $\nu X \to Y(X)$ is a connective cover.

2) $\tau^{-1}\nu X \xrightarrow{\simeq} Y(X)$.

Proof

1)
$$\begin{tikzcd} \nu X \arrow[r] & Y(X) \arrow[r] & C \\ (Syn_E)_{\geq 0} \arrow[u, phantom, "\in" rotate=90] & & (Syn_E)_{\leq -1} \arrow[u, phantom, "\in ?"] \end{tikzcd}$$

But $F(P,X)_{\geq 0} \to F(P,X) \to F(P,X)_{\leq -1}$.

2)
$$\begin{tikzcd} Y(X)(\Sigma P) \arrow[r] \arrow[d, phantom, "\wr"] & \Omega Y(X)(P) \arrow[d, phantom, "\wr"] \\ F(\Sigma P, X) \arrow[r, "\simeq"] & \Omega F(X,P). \end{tikzcd}$$

$$\begin{tikzcd} (\Sigma^{0,-1}X)(P) \arrow[r] \arrow[d, phantom, "\wr"] & (\Sigma^{-1,-1}X)(P) \arrow[d, phantom, "\wr"] \\ X(\Sigma P) \arrow[r] & \Omega X(P) \end{tikzcd}$$

Done since $\tau^{-1}C = 0$. ∎

Theorem $Sp \xrightarrow[\simeq]{Y} Syn_E(\tau^{-1})$

is a symm. monoidal eq.

Proof • For fully faithfulness, reduce to checking

$$\begin{tikzcd} \Omega^\infty F(S^n, X) = map(S^n, X) \arrow[r, phantom, "\overset{??}{\simeq}"] \arrow[d, phantom, "\wr"] & map(Y(S^n), Y(X)) \arrow[d, phantom, "\wr"] \\ \Omega^\infty Y(X)(S^n) \arrow[r, phantom, "\simeq"] & map(\nu S^n, Y(X)) \end{tikzcd}$$

- For essential surjectivity, use that there is a right adjoint $R : \mathrm{Syn}_E(\tau^{-1}) \to \mathrm{Sp}$ for formal reasons, and that it suffices to check $RX \simeq 0 \Rightarrow X = 0$.

$$\begin{aligned} \underset{\substack{\wr\\ *}}{\mathrm{map}(P, RX)} &\simeq \mathrm{map}(y(P), X) \\ &\simeq \mathrm{map}(\nu P, X) \\ &\simeq \Omega^\infty X(P) \end{aligned}$$

$\forall P \in \mathrm{Sp}_E^{fp}$.

$X \in (\mathrm{Syn}_E)_{\leq -1} \Rightarrow X \simeq \tau^{-1}X \simeq 0$. ∎

__Corollary__ $\nu : \mathrm{Sp} \to \mathrm{Syn}_E$

is a fully faithful embedding.

__Proof__

$$\begin{tikzcd} \mathrm{map}(A,B) \arrow[r] \arrow[rr, bend left, "\simeq"] & \mathrm{map}(\nu A, \nu B) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{map}(y(A), y(B)) \arrow[dl, "\simeq"] \\ & \mathrm{map}(\nu A, y(B)) & \end{tikzcd}$$ ∎

__Definition__ $X \in \mathrm{Syn}_E$, let $\tau^{-1}X \in \mathrm{Sp}$ be the underlying spectrum of X.

__Corollary__ $\tau^{-1}\nu X \simeq X$.

__Proof__ $\mathrm{Syn}_E \rightleftarrows \mathrm{Syn}_E(\tau^{-1}) \underset{y}{\overset{\simeq}{\rightleftarrows}} \mathrm{Sp}$

$\nu X \rightsquigarrow \tau^{-1}\nu X = y(X) \mapsto X$. ∎

__Example__ $\tau^{-1} S^{t,w}$

$= \tau^{-1}(\Sigma^{t-w} \nu S^w)$

$= \Sigma^{t-w} \tau^{-1} \nu S^w \simeq S^t$.

4. Modules over $C\tau$

Definition $\mathrm{Stable}_{E_*E} := \mathrm{Shv}_\Sigma^{\mathrm{Sp}}(\mathrm{Comod}^{\mathrm{fp}}_{E_*E})$

Hovey's stable category of comodules.

Lemma

1) $\mathrm{Sp}_E^{\mathrm{fp}} \xrightarrow{E_*} \mathrm{Comod}^{\mathrm{fp}}_{E_*E}$

induces an adjunction

$$\begin{tikzcd}
\epsilon_* : \mathrm{Shv}_\Sigma^{\mathrm{Sp}}(\mathrm{Sp}_E^{\mathrm{fp}}) \arrow[r, shift left] \arrow[d, phantom, "\rotatebox{90}{$=$}"] & \mathrm{Shv}_\Sigma^{\mathrm{Sp}}(\mathrm{Comod}_{E_*E}) : \epsilon^* \arrow[l, shift left] \arrow[d, phantom, "\rotatebox{90}{$=$}"] \\
\mathrm{Syn}_E & \mathrm{Comod}_{E_*E}
\end{tikzcd}$$

2) ϵ_* symm. monoidal, ϵ^* lax symm. monoidal.

3) The diagram

$$\begin{tikzcd}
\mathrm{Sp}_E^{\mathrm{fp}} \arrow[r, "E_*"] \arrow[dd, "\nu|_{\mathrm{Sp}_E^{\mathrm{fp}}}"'] & \mathrm{Comod}^{\mathrm{fp}}_{E_*E} \arrow[d, phantom, "\cap"] \\
 & \mathrm{Stable}^\heartsuit_{E_*E} \arrow[d] \\
\mathrm{Syn}_E \arrow[r, "\epsilon_*"] & \mathrm{Stable}_{E_*E}
\end{tikzcd}$$

Commutes.

Lemma Let $P \in \mathrm{Sp}_E^{\mathrm{fp}}$.

$$C\tau \otimes \nu X \simeq \underset{\mathrm{Stable}^\heartsuit_{E_*E}}{\underset{\cap}{\epsilon^*(E_*P)}}$$

Proof There is a cofiber seq.

$$\Sigma^{-1,0}\nu P \to \nu P \to C\tau \otimes \nu P \simeq (\nu P)_{\leq 0}$$

$(\pi_t^\heartsuit \nu P)_*$ is $(\nu E)_{t+*,*}(\nu P)$ is $E_{t+*}(P)$

The adjunction unit

$$\nu P \to \epsilon^*(\epsilon_* \nu P) \simeq \epsilon^*(E_*P)$$

is also a 0-truncation. $= (\pi_0^\heartsuit P)_*$

__Corollary__ $C\tau$ is a comm. alg.

__Proof__ $C\tau = \varepsilon^*(E_*)$ + ε^* lax monoidal.

__Theorem__

$$\begin{tikzcd} \mathrm{Stable}_{E_*E} \arrow[r, "\varepsilon^*"] \arrow[dr, dashed, "\chi^*"'] & \mathrm{Syn}_E \\ & \mathrm{Mod}_{C\tau} \arrow[u] \end{tikzcd}$$

χ^* has a fully faithful left adjoint χ_*.

__Proof__ All objects of Stable_{E_*E} are E_*-modules $\Rightarrow$ ε^* lands in $\varepsilon^*(E_*)$-modules.
$\varepsilon^*(E_*)$ is $C\tau$. ☒

__Definition__ E has plenty of finite projectives if $\{E_*P \mid P \in Sp_E^{fp}\}$ generate Stable_{E_*E} under colimits.

(Ok for $\mathbb{S}$ and anything Landweber exact.)

__Proposition__ $\chi_* : \mathrm{Mod}_{C\tau} \rightleftarrows \mathrm{Stable}_{E_*E} : \chi^*$ is an eq. $\Leftrightarrow$ E has plenty of finite projectives.

__Proof__ We "know" that χ_* is fully faithful.
$\mathrm{Mod}_{C\tau}$ is generated by modules of the form
$C\tau \otimes \nu P \to \chi_*(C\tau \otimes \nu P) = \varepsilon_*(\nu P) = E_*P$. ☒